The LSD method along sums of three squares

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The Landau–Selberg–Delange (LSD) Method

Classical LSD - asymptotics for sums of divisor bounded multiplicative functions:

$$\sum_{n \leq x} f(n) \sim cx(\log x)^a$$

Question

Asymptotics for

$$N(B) := \sum_{\substack{\mathbf{x} \in \mathbb{Z}^3_{\mathrm{prim}} \\ \|\mathbf{x}\| \leq B}} f(x_1^2 + x_2^2 + x_3^2) \quad ?$$

Comes from problems in arithmetic geometry.

The Classical LSD Method

Theorem (Granville–Koukoulopoulos, 2017)

$f : \mathbb{N} \to \mathbb{C}$ be multiplicative with $|f(n)| \leq \tau_k(n)$ for some $k \in \mathbb{N}$ and $\exists\, \kappa \in \mathbb{C}, x_0 \in \mathbb{R}$ such that

$$\sum_{p \leq x} f(p) \log p = \kappa x + O(x/(\log x)^A) \quad \forall A > 0, x > x_0$$

then

$$\sum_{n \leq x} f(n) \sim cx(\log x)^{\kappa - 1},$$

where c is an Euler product.

Representations as a sum of 3 squares

$$N(B) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^3_{\mathrm{prim}} \\ x_1^2+x_2^2+x_3^2\le B}} f(x_1^2+x_2^2+x_3^2) = \sum_{n\le B} f(n) r_3(n),$$

where $r_3(n) := \#\{\mathbf{x} \in \mathbb{Z}^3_{\mathrm{prim}} : x_1^2+x_2^2+x_3^2 = n\}$.

Theorem (Gauss, Dirichlet, Bateman)

$$r_3(n) = \frac{G_n}{\pi}\sqrt{n} L\left(1, \chi_{-4n}\right),$$

where

$$\chi_{-4n}(m) := \left(\frac{-4n}{m}\right), \quad G_n := \begin{cases} 0 \text{ if } n \equiv 0,4,7 \mod 8 \\ 16 \text{ if } n \equiv 3 \mod 8 \\ 24 \text{ if } n \equiv 1,2,5,6 \mod 8 \end{cases}$$

Strategy

$$N(B) \approx \sqrt{B} \sum_{n \le B} f(n) G_n L(1, \chi_{-4n}).$$

$$N(B) \approx \sqrt{B} \sum_{n \le B} f(n) G_n \sum_{m=1}^{\infty} \frac{\chi_{-4n}(m)}{m}.$$

Split the *m*-sum into 3 ranges:

1. $m > \sqrt{B}(\log B)^{C_1}$, error term (Polya–Vinogradov).
2. $(\log B)^{C_2} < m \le \sqrt{B}(\log B)^{C_1}$, error term (Heath-Brown's large sieve for quadratic characters).
3. $m \le (\log B)^{C_2}$, LSD for $\sum_n f(n) \left(\frac{n}{m}\right)$.

The main term

Truncate at $m \leq (\log B)^{C_2}$:

$$\sum_{n \leq B} \sum_{m \leq (\log B)^{C_2}} G_n f(n) \frac{\chi_{-4n}(m)}{m} = \sum_{m \leq (\log B)^{C_2}} \frac{1}{m} \left(\frac{-4}{m} \right) \sum_{n \leq B} G_n f(n) \left(\frac{n}{m} \right)$$

- G_n periodic mod 8.
- Split inner sum into congruences mod 8.
- Expand congruence condition with Dirichlet characters mod 8.

$\implies$ Use LSD method to find asymptotics for

$$S_{\chi,m}(B) = \sum_{n \leq B} f(n)\chi(n) \left(\frac{n}{m} \right), \text{ for all } \chi \bmod 8, \ m \text{ odd}.$$

Example: Counting soluble conics

Example

Let $f(n)$ be the indicator function of n being the sum of two squares. Equivalently, of the conic $y_1^2 + y_2^2 = ny_3^2$ having a $\mathbb{Q}$-point.

Apply LSD to $f(n)\chi(n)\left(\frac{n}{m}\right)$ for each χ modulo 8 and each odd m. Need the average over the primes. On the primes, f is given by

$$f(p) = \begin{cases} 1 \text{ if } p \equiv 1, 2 \mod 4 \\ 0 \text{ if } p \equiv 3 \mod 4. \end{cases}$$

$\implies$ If $\chi(p)\left(\frac{p}{m}\right) = 1$ for all $p \equiv 1 \mod 4$, have a large 'correlation' with f.

Example: Counting soluble conics (continued)

In fact,

$$\sum_{p \le x} f(p)\chi(p)\left(\frac{p}{m}\right)\log p = \kappa x + O(x/(\log x)^A), \quad \forall A > 0,$$

with

$$\kappa = \begin{cases} \frac{1}{2} \text{ if } \chi = \chi_0 \text{ or } \left(\frac{-4}{\cdot}\right), \text{ and } m = \square \\ 0 \text{ otherwise.} \end{cases}$$

The characters where $\kappa = 1/2$ give a main term $S_{\chi,m}(B) \sim c_{\chi,m}B/(\log B)^{1/2}$ and the characters where $\kappa = 0$ give an error term.

Each main term (χ, m) gives its own Euler product $c_{\chi,m}$.

Example: Counting soluble conics (continued)

Summing over Euler products $\implies$ asymptotic for

$$N(B) := \#\left\{ \begin{smallmatrix} (x_1,x_2,x_3)\in\mathbb{Z}^3_{\mathrm{prim}} : x_1^2+x_2^2+x_3^2\le B, \\ y_1^2+y_2^2=(x_1^2+x_2^2+x_3^2)y_3^2 \text{ has a } \mathbb{Q}\text{-point} \end{smallmatrix} \right\}.$$

Theorem (S. 2026)

$$N(B) \sim \frac{12\prod_p \tau_p}{\pi^{3/2}} \frac{B^{\frac{3}{2}}}{(\log B)^{1/2}},$$

where

$$\tau_p = \left(1-\frac{1}{p}\right)^{1/2} \left(\frac{p^2+\frac{1}{2}\left(1+\left(\frac{-4}{p}\right)\right)p+1}{p^2-1} \right).$$

This is a special case of the Loughran–Smeets conjecture.

Concluding remarks

1. Classical LSD gives
$$\sum_{n\leq x} f(n) \sim cx(\log x)^{\kappa-1},$$
c an Euler product.
2. For many 'LSD multiplicative' functions we proved
$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^3_{\text{prim}}\\ x_1^2+x_2^2+x_3^2\leq B}} f(x_1^2+x_2^2+x_3^2) \sim cB^{3/2}(\log B)^{\kappa-1},$$
3. c is a finite sum of Euler products.
4. Q an integral positive definite ternary quadratic form, f 'LSD multiplicative'. Similar methods $\implies$ asymptotics for
$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^3_{\text{prim}}\\ Q(\mathbf{x})\leq B}} f(Q(\mathbf{x})).$$