Siegel Modular Forms and Theta Series

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Declaration

This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

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Finally, I would like to give special thanks to my parents, for all of their wonderful support over the years which has allowed me to get to this point.

“There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and. . . modular forms.”

A quote often attributed to the German Mathematician Martin Eichler.

Contents

Chapter 1

Introduction

1.1 History

Modular forms rose to fame in the late 20th century. 'Fermat's Last Theorem' was perhaps the most famous unsolved problem in Mathematics up until it was proven in 1994.

The proof of Fermat's Last Theorem largely relied on Andrew Wiles and Richard Taylor's proof of a special case of what is now known as the 'Modularity Theorem'. This theorem built a bridge between 'Elliptic curves' and 'Modular Forms', and thus thrust the study of modular forms into the public eye.

However, modular forms and their generalisations have been studied since far before the time of Wiles or Taylor. Indeed mathematicians have been considering modular forms in some way since the early 19th century [8].

In the 1930s and 1940s the mathematician Carl Ludwig Siegel adapted the theory of classical modular forms for the purposes of studying quadratic forms [13]. The new type of modular forms he devised are now known eponymously.

1.2 Motivation

The study of classical theta series is motivated by the question of understanding a type of arithmetic object called the 'Representation numbers':

Definition 1. *Take F to be a positive definite, integral matrix with even entries on the diagonal. The representation number [10] of $g \in \mathbb{N}$ by F is defined as the number of ways to represent g by F:*

$$r_F(g) := \#\{\mathbf{x} \in \mathbb{Z}^m : \frac{1}{2}\,{}^t\mathbf{x}F\mathbf{x} = g\} \tag{1.1}$$

We can use a type of modular form called a theta series to answer this question.

Definition 2. *The theta series associated with F is:*

$$\theta_F(\tau) := \sum_{\mathbf{x} \in \mathbb{Z}^m} e^{\pi i ({}^t\mathbf{x}F\mathbf{x})\tau} = \sum_{g=0}^{\infty} r_F(g) q^g \tag{1.2}$$

On the right we have let $q = e^{2\pi i \tau}$ and rearranged the series to find that its Fourier coefficients are the representation numbers (1.1). We can then find bounds or explicit formulae for the $r_F(g)$ by applying results from the theory of modular forms. [12]

A natural question to ask is whether we can find some automorphic forms with the following, more general, representation numbers as Fourier coefficients:

Definition 3. *Let $G \in \mathrm{M_n}(\mathbb{Z})$ with even entries on the diagonal, F as defined above. The representation number [5] of G by F is:*

$$r_F(G) \colon = \#\{M \in \mathrm{M}_{m\times n}(\mathbb{Z}) \colon {}^tMFM = G\} \tag{1.3}$$

Siegel modular forms are the correct generalisation of classical modular forms to provide us with an answer to this question.

1.3 Outline of this report

In this report we shall explore Siegel modular forms and their theta series.

We will spend the first half of this report discussing many of the important notions which underpin the theory. We define an analogue of $\mathrm{SL}_2(\mathbb{R})$ called the 'Symplectic Group' and a space for it to act on: 'Siegel's half-space'. We will define a group action for this pair and prove it is well-defined.

We shall then define Siegel's modular group and a number of other useful discrete groups and investigate the fundamental domain. We will then be able to state the definition of a Siegel modular form and discuss some properties and differences to the classical case.

In particular we will discuss the 'Koecher principle' which is the major difference between classical modular forms and Siegel modular forms more generally.

Finally, in the second half of this report we will define 'Siegel theta series' which are an analogue of our familiar theta series from the classical theory of modular forms. We will then prove that these Siegel theta series are indeed Siegel modular forms which transform with a particular character. The final end goal of the report will be to determine the exact character for some particular cases of Siegel theta series.

1.4 The Literature

This report is mostly based on well established theory and thus we will draw from a plethora of papers, lecture notes, websites and textbooks. The report focusses on filling in any gaps of proofs not documented in the literature, as well as drawing links between texts where comparisons can be made in the different approaches made by the various authors.

Chapters 2, 3 and 5 are primarily based on the book of Klingen [17]. Chapter 4 loosely follows the book of Maass [22] whilst also drawing from Klingen. In chapter 6 we will draw from the notes of Kohnen [19] as well as referring to the PhD thesis of Dickson [10]. Finally, chapter 7 is primarily based on a paper by Andrianov and Maloletkin [5].

Of course, in each chapter we will also draw from plenty of other sources including the book of Andrianov and Zhuralev [6], the book of Eichler [11], the notes of Van Der Geer [13] and the notes of Funke from the Durham University level IV course on Modular Forms [12]. There are also a number of other sources used for smaller results which can be found in the Bibliography.

Throughout this report we will use a consistent notation which combines the best aspects of the various conventions which appear throughout the literature. All notation will be properly introduced at the necessary points, but there is also a glossary of notation in Appendix A which contains a list of all notation and conventions used throughout this report.

Chapter 2

The Symplectic Group and Siegel's half-space

In order to study Siegel modular forms, we must first set up our 'universe' of Siegel modular forms. Siegel modular forms are a type of automorphic forms. Therefore they must be defined on some space and have a transformation property under a group of automorphisms acting on that space. In this section we will define a space which will act as the domain for our Siegel modular forms, as well as a group of automorphisms on this space.

Most of this section follows the book of Klingen titled 'Introductory lectures on Siegel modular forms' [17], however we will adjust the order of content to better suit our purposes, as well as fleshing out most of the proofs with full justifications which are otherwise missing from Klingen's book.

We will also refer to the book of Maass [22] titled 'Siegel's Modular Forms and Dirichlet Series' which takes a much wider look at the theory and is occasionally useful for highlighting details which Klingen misses.

2.1 Siegel's half-space

We should first recall what it means for a matrix to be positive definite.

Definition 4. *A symmetric real $n \times n$ matrix y is positive definite if for all $\mathbf{v} \in \mathbb{R}^n$, ${}^t\mathbf{v}y\mathbf{v} \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$.*

We note that, in this report, all positive definite matrices are assumed to be symmetric.

We will frequently use the shorthand $y > 0$ to mean that y is positive definite. If we relax the 'if and only if' condition to allow for ${}^t\mathbf{v}y\mathbf{v} = 0$ when $\mathbf{v} \neq \mathbf{0}$ we call the matrix 'positve semi-definite' and denote this by $y \geq 0$.

We are now ready to define Siegel's half-space, as per Klingen's book [17].

Definition 5. *Siegel's half-space is defined to be the space of all $n \times n$ complex symmetric matrices with imaginary part positive definite.*

$$\mathbb{H}_n := \{z = x + iy : z \in \mathrm{M}_n(\mathbb{C}), z = {}^tz, y > 0\} \tag{2.1}$$

2.1.1 Siegel's half-space as a subset of a complex vector space

We can assign a 'coordinate system' to $\mathbb{H}_n$. Matrices in $\mathbb{H}_n$ have independent entries z_{kl} for $k \leq l$, since the other entries are fixed by the symmetry condition. Thus we can view the z_{kl} as coordinates of Siegel's half-space. There are $\frac{n(n+1)}{2}$ independent entries since this is the nth triangle number, so our space is $\frac{n(n+1)}{2}$-dimensional.

As a result, we can view $\mathbb{H}_n$ as an open subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$.

Proposition 6. *$\mathbb{H}_n$ is a convex subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$. That is, any two points in $\mathbb{H}_n$ can be joined by a straight line, with the line lying within $\mathbb{H}_n$.*

Klingen asserts this fact in his book, but we shall go more in depth in justifying the proof.

Proof. For arbitrary $z_1, z_2 \in \mathbb{H}_n$, define the path

$$\begin{aligned} f : [0,1] &\rightarrow \mathbb{H}_n \\ \lambda &\mapsto \lambda z_1 + (1-\lambda) z_2 = f(\lambda) \end{aligned} \tag{2.2}$$

Firstly, we see that this is a straight line joining z_1 and z_2 because:

$$\begin{aligned} f(0) &= z_2 \\ f(1) &= z_1 \\ f(\lambda) &= \lambda(z_1 - z_2) + z_2 \end{aligned} \tag{2.3}$$

(2.3) is clearly in the form $y = mx + c$ for a straight line.

Next we need to check that $f(\lambda) \in \mathbb{H}_n \forall \lambda \in [0,1]$. Recall that elements of $\mathbb{H}_n$ need to be symmetric and have positive definite imaginary part.

Symmetry is clear, because for all values of λ, $f(\lambda)$ is a linear combination of z_1 and z_2 which are symmetric matrices. Thus ${}^t(f(\lambda)) = \lambda\, {}^t z_1 + (1-\lambda)\, {}^t z_2 = \lambda z_1 + (1-\lambda) z_2 = f(\lambda)$.

$\mathrm{Im}(f(\lambda))$ being positive definite is less obvious. Recall that we require ${}^t\mathbf{v}(\mathrm{Im} f(\lambda))\mathbf{v} > 0$ for all non-zero $\mathbf{v} \in \mathbb{R}^n$ and equal to 0 for $\mathbf{v} = \mathbf{0}$.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, and define $g(\lambda) := \mathrm{Im}(f(\lambda))$

Take an arbitrary non-zero $\mathbf{v} \in \mathbb{R}$.

Then ${}^t\mathbf{v} g(\lambda)\mathbf{v} = {}^t\mathbf{v}(\lambda y_1 + (1-\lambda)y_2)\mathbf{v} = \lambda\, {}^t\mathbf{v} y_1 \mathbf{v} + (1-\lambda)\, {}^t\mathbf{v} y_2 \mathbf{v} > 0$

We can deduce the last inequality because λ and $1-\lambda$ are both positive for $\lambda \in [0,1]$ and y_1, y_2 are both positive definite. Clearly the case where $\mathbf{v} = \mathbf{0}$ trivially gives equality.

So the imaginary part of $f(\lambda)$ is positive definite, and $f(\lambda)$ is symmetric. Therefore, we deduce that $f(\lambda)$ is within $\mathbb{H}_n$ for all $\lambda \in [0,1]$. So the straight line joining z_1 and z_2 is contained within $\mathbb{H}_n$, and since z_1 and z_2 were arbitrary this is true for any two points in $\mathbb{H}_n$. Thus $\mathbb{H}_n$ is a convex subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$. □

2.1.2 Low values of n

It is fairly clear to see that for $n = 1$, Siegel's half-space $\mathbb{H}_1$ coincides with our standard definition of the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

However, for $n > 1$ it is more difficult to visualise Siegel's half-space. As we saw before, $\mathbb{H}_n$ is a subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$. The problem is that the number of dimensions increases rapidly with n.

For $n = 2$ we see that $\mathbb{H}_2 \subset \mathbb{C}^3$ which is a three dimensional complex space. Our universe is real though - so if we wanted to make a drawing or plot of this space we would need 6 real dimensions.

There is a clever way of being able to visualise $\mathbb{H}_2$ however. Firstly we split $\mathbb{H}_n$ into real and imaginary parts which we will consider separately.

$$\begin{aligned} \mathbb{H}_n^{\mathrm{Re}} &:= \{x : x \in \mathrm{M}_n(\mathbb{R}),\ x = {}^t x\} \\ \mathbb{H}_n^{\mathrm{Im}} &:= \{y : y \in \mathrm{M}_n(\mathbb{R}),\ y = {}^t y,\ y > 0\} \end{aligned} \tag{2.4}$$

It is clear to see how matrices $z \in \mathbb{H}_n$ are constructed by $z = x + iy$ for some $x \in \mathbb{H}_n^{\mathrm{Re}}$ and $y \in \mathbb{H}_n^{\mathrm{Im}}$, so indeed $\mathbb{H}_n = \mathbb{H}_n^{\mathrm{Re}} \oplus i\mathbb{H}_n^{\mathrm{Im}}$.

The following lemma is stated without proof in section 2 of Klingen's book [17].

Lemma 7. $\mathbb{H}_n^{\mathrm{Im}}$ *is a convex cone with vertex at the origin. That is, every ray originating from the point* 0 *and passing through a point* $y \in \mathbb{H}_n^{\mathrm{Im}}$ *lies entirely within* $\mathbb{H}_n^{\mathrm{Im}}$*.*

Proof. We can parametrise a ray originating from 0 passing through a point $y \in \mathbb{H}_n^{\mathrm{Im}}$ with the following function:

$$\begin{aligned} f : (0, \infty) &\rightarrow \mathbb{H}_n^{\mathrm{Im}} \\ t &\mapsto ty \end{aligned} \tag{2.5}$$

This ray clearly originates from 0 since $f(0) = 0$ and passes through y since $f(1) = y$.

Every point on the ray is of the form $f(t) = ty$. Clearly since $t > 0$ and y is positive definite, this implies that the matrix ty is positive definite. Hence the point ty lies in $\mathbb{H}_n^{\mathrm{Im}}$ □

We will now see how to visualise $\mathbb{H}_2$

Let us first consider $x \in \mathbb{H}_2^{\mathrm{Re}}$. What does the matrix x look like? Using the symmetry condition we are able to fix one of the entries on the off diagonal, meaning a generic x looks like:

$$x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \cong \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \quad ; \quad x_1, x_2, x_3 \in \mathbb{R} \tag{2.6}$$

So we see that $\mathbb{H}_2^{\mathrm{Re}}$ can be viewed as a subset of $\mathbb{R}^3$. Moreover, because there are no further constraints on the entries x_1, x_2, x_3 beyond the symmetry condition, we see that $\mathbb{H}_2^{\mathrm{Re}} \cong \mathbb{R}^3$ as vector spaces. Indeed $\mathbb{H}_2^{\mathrm{Re}}$ is actually rather trivial to consider.

Now we move onto $\mathbb{H}_2^{\mathrm{Im}}$ which is much more interesting. Let's consider a generic matrix $y \in \mathbb{H}_2^{\mathrm{Im}}$. We can apply the symmetry condition as before to fix one of the entries on the off diagonal, but we also have the positive definite condition to consider.

$$\begin{aligned} y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} &\cong \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \quad ; \quad y_1, y_2, y_3 \in \mathbb{R} \\ y_1 > 0 \quad &; \quad y_1 y_3 - y_2^2 > 0 \end{aligned} \tag{2.7}$$

The second line here is the result of the positive definite condition. This arises from Sylvester's Criterion [20].

We once again notice that $\mathbb{H}_2^{\mathrm{Im}}$ can be viewed as a subset of $\mathbb{R}^3$, but this time we have a more interesting subset due to the restrictions on our coordinates y_1, y_2, y_3.

We can plot this subset of $\mathbb{R}^3$ using graphing software. Such a plot can be seen in figure 2.1.

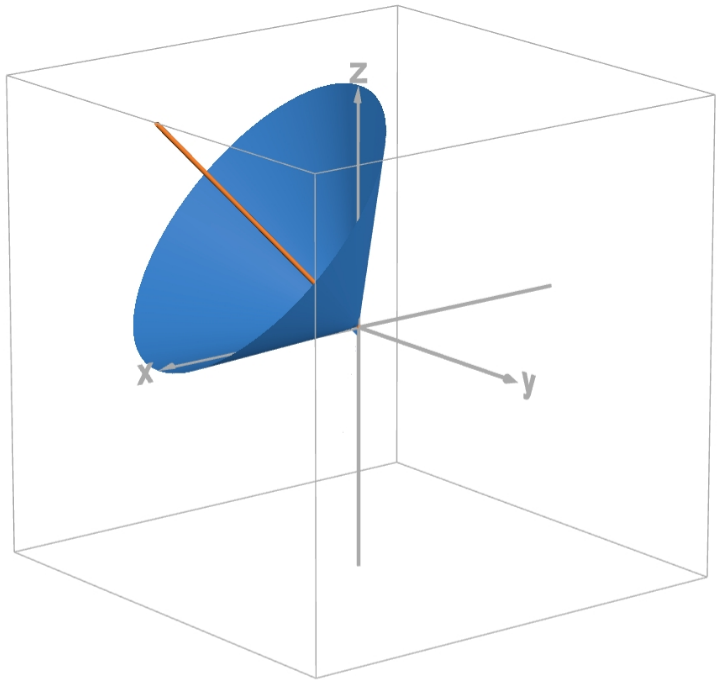

Figure 2.1: The blue surface is the boundary of the imaginary part of $\mathbb{H}_2$. The orange line is the 'axis' of the 'cone'.

2.2 The Symplectic Group

We will now define a group to act on $\mathbb{H}_n$.

Definition 8. *Let $n \in \mathbb{N}$. The symplectic group of degree n over $\mathbb{R}$ is*

$$\mathrm{Sp}(n, \mathbb{R}) := \{\gamma \in \mathrm{GL}(2n, \mathbb{R}) : j[\gamma] = j\} \tag{2.8}$$

where

$$j = \begin{pmatrix} 0_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0_{n\times n} \end{pmatrix}, \text{ and } j[\gamma] := {}^t\gamma j \gamma \tag{2.9}$$

Note that the symplectic group of order n is a group of $2n \times 2n$ matrices. We often write such matrices in $n \times n$ blocks, as we have done with j. We call elements of the symplectic group 'symplectic matrices' and write them in $n \times n$ blocks as $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It should be fairly quick to see that $\mathrm{Sp}(n, \mathbb{R})$ is a group. The proof can be found in Appendix C, as well as some justification as to why this group is meaningful.

2.2.1 Characterising the Symplectic Group

We might ask about the determinant of matrices in the symplectic group. It is elementary to see that for all $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ we have $\det(\gamma) = \pm 1$. This follows directly from taking determinants in the condition ${}^t\gamma j \gamma = j$.

However, Maass [22] further deduces in chapter 4 of his book that $\det(\gamma) = 1$. We shall cover this proof later once we have introduced the group action.

Klingen [17] states a set of conditions which we can use to characterise elements of the symplectic group, although he does not detail any steps of the derivation. We shall discuss the derivation.

As mentioned before, it is often natural to decompose matrices in the symplectic group into $n \times n$ blocks

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.10}$$

where $a, b, c, d \in \mathrm{M}_n(\mathbb{R})$. In this form, we can apply the condition $j[\gamma] = j$ and find conditions on the a, b, c, d which fully characterise symplectic matrices.

Writing the matrices γ and j in block form we see that our condition is equivalent to

$$\begin{pmatrix} {}^ta & {}^tc \\ {}^tb & {}^td \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.11}$$

Now we multiply out the left hand side to get

$$\begin{pmatrix} {}^tac - {}^tca & {}^tad - {}^tcb \\ {}^tbc - {}^tda & {}^tbd - {}^tdb \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{2.12}$$

Comparing the 4 different $n \times n$ blocks gives us 4 equations

$$\begin{aligned} {}^tac &= {}^tca & (2.13) \\ {}^tbd &= {}^tdb & (2.14) \\ {}^tad - {}^tcb &= \mathbb{1} & (2.15) \\ {}^tda - {}^tbc &= \mathbb{1} & (2.16) \end{aligned}$$

although notice that the last two equations are just transposes of each other, and are therefore equivalent. We thus have three conditions which fully classify all matrices in the symplectic group:

$$\begin{aligned} &{}^tac \quad \text{symmetric} & (2.17) \\ &{}^tbd \quad \text{symmetric} & (2.18) \\ &{}^tad - {}^tcb = \mathbb{1} & (2.19) \end{aligned}$$

Klingen [17] asserts that we can apply the same logic to $j[{}^t\gamma] = j$ to get three different but equivalent conditions to separately characterise symplectic matrices

$$\begin{aligned} &a\,{}^tb \quad \text{symmetric} & (2.20) \\ &c\,{}^td \quad \text{symmetric} & (2.21) \\ &a\,{}^td - b\,{}^tc = \mathbb{1} & (2.22) \end{aligned}$$

These conditions will be referred to as 'symplecticity conditions'. If we refer to 'symplecticity' during a proof then it means we are invoking one or more of these conditions.

Chapter 3

The group action

3.1 The classical group action

In order to motivate our definition of the group action we shall first take a step back and consider the structure of $\mathrm{Sp}(1, \mathbb{R})$. We know that every matrix in the symplectic group can be characterised by the three symplecticity conditions (2.19). When $n = 1$ we have that a, b, c, d are just real numbers. Therefore the symmetry conditions are automatically satisfied, and the third equation just becomes $\det(\gamma) = 1$.

Therefore, the symplectic group of degree 1 is just the group of 2×2 real matrices with determinant 1. We have a name for this group: The Special Linear group $\mathrm{SL}_2(\mathbb{R})$.

Recall from complex analysis [21] that $\mathrm{SL}_2(\mathbb{R})$ acts on the complex upper half-plane via Möbius transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad \tau \in \mathbb{H}$$

$$\gamma\langle\tau\rangle := \frac{a\tau + b}{c\tau + d} \tag{3.1}$$

We saw from our discussion of $\mathbb{H}_n$ for low values of n that $\mathbb{H}_1 = \mathbb{H}$, so we would like to define an action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathbb{H}_n$ that naturally generalises the above action for $n > 1$.

3.2 Holomorphic functions of multiple variables

We must briefly introduce the idea of holomorphicity for complex functions of multiple variables. We shall take the definition from the book of Gunning and Rossi [14].

Definition 9. *A complex-valued function f defined on an open subset $D \subset \mathbb{C}^n$ is called holomorphic in D if each point $w = (w_1, w_2, ..., w_n) \in D$ has an open neighbourhood U with $w \in U \subset D$ such that the function f has a power series expansion which converges for all $z = (z_1, z_2, ..., z_n) \in U$:*

$$f(z) = \sum_{v_1, v_2, ..., v_n = 0}^{\infty} a_{v_1, v_2, ..., v_n} (z_1 - w_1)^{v_1} (z_2 - w_2)^{v_2} ... (z_n - w_n)^{v_n} \tag{3.2}$$

This is clearly analogous to the one variable case. We also have the following result known as Osgood's lemma which is also stated (and proven) in the book of Gunning and Rossi [14]. We will not prove it here to avoid getting sidetracked from our main discussion.

Theorem 10. *If a complex-valued function f is continuous on an open subset $D \subset \mathbb{C}^n$, and is holomorphic in each variable separately, then it is holomorphic in D.*

As per Knill's notes on "A short introduction to several complex variables" [18], this idea can be extended further to functions from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Definition 11. *Given $D_1 \in \mathbb{C}^n$ an open subset and a function $f : D_1 \to \mathbb{C}^m$, f is called holomorphic if each of the coordinate functions $f_k : D_1 \to \mathbb{C}$ is holomorphic. If $D_2 \subset \mathbb{C}^m$ an open subset we call $f : D_1 \to D_2$ biholomorphic if there is an inverse function $f^{-1} : D_2 \to D_1$ such that f^{-1} is holomorphic.*

3.3 The Symplectic Group acting on Siegel's half-space

We can define a group action of the symplectic group on Siegel's half-space, noting that when $n = 1$ we recover the case of Möbius transformations. [17]

Proposition 12. $\mathrm{Sp}(n, \mathbb{R})$ *acts on $\mathbb{H}_n$ as a group of biholomorphic automorphisms by:*

$$\begin{aligned} \mathrm{Sp}(n, \mathbb{R}) \times \mathbb{H}_n &\to \mathbb{H}_n \\ (\gamma, z) &\mapsto \gamma\langle z\rangle := (az+b)(cz+d)^{-1} \end{aligned} \tag{3.3}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.4}$$

We will prove Proposition 12. This proof closely follows the proof on pages 2 and 3 of Klingen's book [17], with the missing details filled in.

First of all, we need to show that the action is well defined, so we begin by proving the following lemma:

Lemma 13. *The matrix $(cz + d)$ is invertible for all $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ and $z \in \mathbb{H}_n$.*

Proof. We set

$$\begin{aligned} p &:= az + b \\ q &:= cz + d \end{aligned}$$

and consider the expression

$$^t p\overline{q} - {}^t q\overline{p} = {}^t(az+b)\overline{(cz+d)} - {}^t(cz+d)\overline{(az+b)} \tag{3.5}$$

Klingen [17] then skips to the conclusion. We shall perform the algebraic heavy lifting:

The expression (3.5) rearranges to

$$^t p\overline{q} - {}^t q\overline{p} = ({}^t z\, {}^t a + {}^t b)(c\overline{z} + d) - ({}^t z\, {}^t c + {}^t d)(a\overline{z} + b) \tag{3.6}$$

since we know that a, b, c, d are all real valued matrices. We also have that z is a symmetric matrix by the definition of $\mathbb{H}_n$, so can deduce that this expression is equal to

$$^t p\overline{q} - {}^t q\overline{p} = (z\, {}^t a + {}^t b)(c\overline{z} + d) - (z\, {}^t c + {}^t d)(a\overline{z} + b) \tag{3.7}$$

Now we expand the brackets and obtain

$$ {}^t p\overline{q} - {}^t q\overline{p} = z\,{}^t ac\overline{z} + {}^t bc\overline{z} + z\,{}^t ad + {}^t bd - (z\,{}^t ca\overline{z} + {}^t da\overline{z} + z\,{}^t cb + {}^t db) \tag{3.8} $$

We apply our symplecticity conditions (2.19) and re-order the terms to get

$$ {}^t p\overline{q} - {}^t q\overline{p} = z\,{}^t({}^t ac)\overline{z} - z\,{}^t ca\overline{z} + {}^t bc\overline{z} - {}^t da\overline{z} + z\,{}^t ad - z\,{}^t cb + {}^t({}^t bd) - {}^t db \tag{3.9} $$

We notice that the first two terms now just cancel with each other once we distribute the transpose, and likewise with the last two terms, which leaves us with

$$ {}^t p\overline{q} - {}^t q\overline{p} = ({}^t bc - {}^t da)\overline{z} + z({}^t ad - {}^t cb) \tag{3.10} $$

We once again apply some symplecticity conditions (2.19) to obtain

$$ {}^t p\overline{q} - {}^t q\overline{p} = -\mathbb{1}\overline{z} + z\mathbb{1} = z - \overline{z} = 2iy \tag{3.11} $$

where y is the imaginary part of z. The formula (3.11) is extremely useful and we will refer back to it on multiple occasions.

We now pick up with Klingen once again and consider an arbitrary column vector $\boldsymbol{\xi} \in \mathbb{C}^n$ such that $q\boldsymbol{\xi} = \mathbf{0}$.

Now we investigate the quantity $y\{\overline{\boldsymbol{\xi}}\}$, where we are using the notation $a\{b\} = {}^t\overline{b}ab$. Klingen does not go into detail on the following justification so we shall take care to go step-by-step,

$$ \begin{aligned} y\{\overline{\boldsymbol{\xi}}\} &= {}^t\boldsymbol{\xi} y \overline{\boldsymbol{\xi}} \\ &= {}^t\boldsymbol{\xi}\frac{1}{2i}({}^t p\overline{q} - {}^t q\overline{p})\overline{\boldsymbol{\xi}} \end{aligned} \tag{3.12} $$

where we have invoked the formula (3.11).

$$ \begin{aligned} y\{\overline{\boldsymbol{\xi}}\} &= \frac{1}{2i}({}^t\boldsymbol{\xi}\,{}^t p\overline{q}\overline{\boldsymbol{\xi}} - {}^t\boldsymbol{\xi}\,{}^t q\overline{p}\overline{\boldsymbol{\xi}}) \\ &= \frac{1}{2i}({}^t\boldsymbol{\xi}\,{}^t p\overline{(q\boldsymbol{\xi})} - {}^t(q\boldsymbol{\xi})\overline{p}\overline{\boldsymbol{\xi}}) \end{aligned} \tag{3.13} $$

and since $q\boldsymbol{\xi} = \mathbf{0}$:

$$ \begin{aligned} y\{\overline{\boldsymbol{\xi}}\} &= \frac{1}{2i}({}^t\boldsymbol{\xi}\,{}^t p\overline{(\mathbf{0})} - {}^t(\mathbf{0})\overline{p}\overline{\boldsymbol{\xi}}) \\ y\{\overline{\boldsymbol{\xi}}\} &= 0 \end{aligned} \tag{3.14} $$

By the definition of $\mathbb{H}_n$, y is a positive definite matrix, which means that

$$ y\{\overline{\boldsymbol{\xi}}\} = 0 \iff \boldsymbol{\xi} = \mathbf{0} \tag{3.15} $$

and so we have deduced that $\boldsymbol{\xi} = \mathbf{0}$ is the only solution to the system $q\boldsymbol{\xi} = \mathbf{0}$. This is exactly the requirement for q to be non-singular and hence invertible. □

So we have proven our lemma that $(cz + d)$ is always invertible, and thus $\gamma\langle z\rangle$ is a well-defined function. We still need to prove Proposition 12: that $\gamma\langle z\rangle$ is indeed a group action on $\mathbb{H}_n$.

Proof. First and foremost we want to show that the image of $z \in \mathbb{H}_n$ under the action of $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ also lies in $\mathbb{H}_n$.

We set $z' := \gamma\langle z \rangle$ to be the image of z under the group action. Note that we use different notation to Klingen who instead uses z^* - we avoid this due to potential confusion with the conjugate transpose.

The real and imaginary parts of z' shall be denoted x' and y' respectively.

We need to check two things to show that $z' \in \mathbb{H}_n$:

1. z' is symmetric. That is, ${}^t z' = z'$
2. y' is positive definite.

We begin with showing the symmetry of z'. Klingen omits the details of this so we will cover them here.

First, note that $z' = \gamma\langle z \rangle = (az + b)(cz + d)^{-1} = pq^{-1}$ using our definitions of p and q from before.

We want to show ${}^t z' = z'$, which can be seen as equivalent to:

$$\begin{aligned} {}^t z' &= z' \\ \iff {}^t(pq^{-1}) &= pq^{-1} \\ \iff {}^t(q^{-1})\,{}^t p &= pq^{-1} \\ \iff {}^t pq &= {}^t qp \end{aligned} \tag{3.16}$$

So we will prove that the equation (3.16) holds. This requires some algebraic manipulation and relies on our symplecticity conditions (2.19).

$${}^t pq = {}^t(az + b)(cz + d) = ({}^t z\,{}^t a + {}^t b)(cz + d) \tag{3.17}$$

Recalling that z is symmetric

$${}^t pq = (z\,{}^t a + {}^t b)(cz + d) = z\,{}^t acz + z\,{}^t ad + {}^t bcz + {}^t bd \tag{3.18}$$

Invoking symplecticity (and implicitely, the symmetry of z)

$$\begin{aligned} {}^t pq &= z\,{}^t caz + z(\mathbb{1} + {}^t cb) + {}^t bcz + {}^t db \\ &= z\,{}^t caz + z\,{}^t cb + z + {}^t(z\,{}^t cb) + {}^t db \\ &= z\,{}^t caz + z\,{}^t cb + z + {}^t(z({}^t ad - \mathbb{1})) + {}^t db \\ &= z\,{}^t caz + z\,{}^t cb + z + ({}^t da - \mathbb{1})z + {}^t db \\ &= z\,{}^t caz + z\,{}^t cb + {}^t daz + {}^t db \\ &= (z\,{}^t c + {}^t d)(az + b) \end{aligned} \tag{3.19}$$

And once again by the symmetry of z

$${}^t pq = ({}^t z\,{}^t c + {}^t d)(az + b) = {}^t qp \tag{3.20}$$

as required. So z' is indeed symmetric. Now we want to show that $y' = \mathrm{Im}(z')$ is positive definite. We start by writing y' in terms of z' in the style of (3.11):

$$y' = \frac{1}{2i}(z' - \overline{z'}) \tag{3.21}$$

Now we use the symmetry of z'

$$\begin{aligned} y' &= \frac{1}{2i}({}^t z' - \overline{z'}) \\ &= \frac{1}{2i}({}^t q^{-1}\,{}^t p - \overline{pq^{-1}}) \end{aligned} \tag{3.22}$$

We now use a subtle trick of introducing a $\overline{qq^{-1}}$ in the first term in order to factor out a $\overline{q}^{-1}$ on the right

$$y' = \frac{1}{2i}\left(({}^t q^{-1}\,{}^t p\overline{q} - \overline{p})\overline{q}^{-1}\right) \tag{3.23}$$

Similarly we now introduce a ${}^t q^{-1}\,{}^t q$ in the second term in order to factor out a ${}^t q^{-1}$ on the left

$$\begin{aligned} y' &= \frac{1}{2i}\left({}^t q^{-1}({}^t p\overline{q} - {}^t q\overline{p})\overline{q}^{-1}\right) \\ &= \frac{1}{2i}({}^t p\overline{q} - {}^t q\overline{p})\{\overline{q}^{-1}\} \end{aligned} \tag{3.24}$$

where we are once again using the notation $a\{b\} = {}^t\overline{b}ab$. Now we apply the formula (3.11) to get:

$$y' = y\{\overline{q}^{-1}\} \tag{3.25}$$

We want to show that y' is positive definite. Take an arbitrary column $\mathbf{v} \in \mathbb{C}^n$ and consider ${}^t\overline{\mathbf{v}}y'\mathbf{v}$.

$${}^t\overline{\mathbf{v}}y'\mathbf{v} = {}^t\overline{\mathbf{v}}(y\{\overline{q}^{-1}\})\mathbf{v} = {}^t\overline{\mathbf{v}}\,{}^t q^{-1}y\overline{q}^{-1}\mathbf{v} = {}^t(q^{-1}\overline{\mathbf{v}})y(\overline{q}^{-1}\mathbf{v}) = y\{\overline{q}^{-1}\mathbf{v}\} \tag{3.26}$$

Since y is real, symmetric and positive definite it is trivially Hermitian. Therefore, we can see that $y\{\overline{q}^{-1}\mathbf{v}\} \geq 0$ for all $\mathbf{v} \in \mathbb{C}^n$ with equality if and only if $\mathbf{v} = \mathbf{0}$.

Hence $y'\{\mathbf{v}\} \geq 0$ for all complex $\mathbf{v}$ and thus $y'[\mathbf{v}] \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ with equality if and only if $\mathbf{v} = \mathbf{0}$. This is exactly the criterion for y' being positive definite.

So z' satisfies the two conditions necessary for it to be in Siegel's half-space, so indeed $\gamma\langle z\rangle \in \mathbb{H}_n$.

It should be fairly easy to convince oneself that the mappings $z \mapsto \gamma\langle z\rangle$ are biholomorphic by the fact that $(cz + d)$ is non-singular for all $z \in \mathbb{H}_n$ and that each component of $\gamma\langle z\rangle$ is just a rational function of the coordinates of $\mathbb{H}_n$. The final things we need to check in order for $\gamma\langle z\rangle$ to define a group action are:

$$\begin{aligned} (\gamma_1\gamma_2)\langle z\rangle &= \gamma_1\langle\gamma_2\langle z\rangle\rangle \\ \mathbb{1}\langle z\rangle &= z \end{aligned} \tag{3.27}$$

The second of these follows immediately from the definition of the group action.

We shall now prove the first of these conditions which requires some basic algebraic manipulation. Klingen leaves this out of his book [17] so we shall write it up for the sake of completeness.

Firstly let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. We see that $\gamma_1\gamma_2 = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$. From the definition of the group action we see that

$$\gamma_1\gamma_2\langle z\rangle = ((ae+bg)z + (af+bh))\left((ce+dg)z + (cf+dh)\right)^{-1} \tag{3.28}$$

Now we compute $\gamma_1\langle\gamma_2\langle z\rangle\rangle$ via the definition of the group action:

$$\begin{aligned} z' &:= \gamma_1\langle\gamma_2\langle z\rangle\rangle = \gamma_1\left\langle (ez+f)(gz+h)^{-1}\right\rangle \\ &= \left(a(ez+f)(gz+h)^{-1}+b\right)\left(c(ez+f)(gz+h)^{-1}+d\right)^{-1} \end{aligned} \tag{3.29}$$

The trick now is to clear the inverses to allow for easier manipulation:

$$\begin{aligned} z'\left(c(ez+f)(gz+h)^{-1}+d\right) &= a(ez+f)(gz+h)^{-1}+b \\ \iff z'\left(c(ez+f)+d(gz+h)\right) &= a(ez+f)+b(gz+h) \\ \iff z'(cez+cf+dgz+dh) &= aez+af+bgz+bh \\ \iff z'\left((ce+dg)z+(cf+dh)\right) &= (ae+bg)z+(af+bh) \end{aligned} \tag{3.30}$$

So we can now get an expression for $\gamma_1\langle\gamma_2\langle z\rangle\rangle$ which we notice matches the expression for $\gamma_1\gamma_2\langle z\rangle$:

$$\begin{aligned} \gamma_1\langle\gamma_2\langle z\rangle\rangle = z' = \left((ae+bg)z+(af+bh)\right)\left((ce+dg)z+(cf+dh)\right)^{-1} \\ = \gamma_1\gamma_2\langle z\rangle \end{aligned} \tag{3.31}$$

□

Now that we have a well defined group action, we will investigate some of its properties and consequences.

3.4 Further properties of the group action

3.4.1 Determinant of Symplectic Matrices

We shall now return to the question posed in section 2.2.1 about the determinant of matrices in $\mathrm{Sp}(n,\mathbb{R})$. Having defined the group action we will be able to say a little more about the determinant of symplectic matrices.

We already found that for all $\gamma \in \mathrm{Sp}(n,\mathbb{R})$, $\det(\gamma) = \pm 1$. Maass [22] uses the following argument to deduce that in fact $\det(\gamma) = 1$.

Lemma 14. $\det(\gamma) = 1$ *for all* $\gamma \in \mathrm{Sp}(n,\mathbb{R})$

Proof. Take an arbitrary $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$ and $z \in \mathbb{H}_n$. Denote $z' = \gamma\langle z\rangle$ and consider the following:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z & \overline{z} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} &= \begin{pmatrix} az+b & a\overline{z}+b \\ cz+d & c\overline{z}+d \end{pmatrix} \\ \gamma\begin{pmatrix} z & \overline{z} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} &= \begin{pmatrix} z' & \overline{z'} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} cz+d & 0 \\ 0 & c\overline{z}+d \end{pmatrix} \end{aligned} \tag{3.32}$$

One can verify that the right hand side in (3.32) rearranges to the right hand side in the above line by writing out the definition of the group action.

Notice that

$$\begin{aligned}
\det\begin{pmatrix} cz+d & 0 \\ 0 & c\overline{z}+d \end{pmatrix} &= \det\left(\begin{pmatrix} cz+d & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & c\overline{z}+d \end{pmatrix}\right) \\
&= \det(cz+d)\det(c\overline{z}+d) \qquad (3.33)
\end{aligned}$$

We now use a trick found in the proof of the 5th proposition in Marco Taboga's lectures on matrix algebra [25] to help us calculate the determinants of $\begin{pmatrix} z & \overline{z} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$ and $\begin{pmatrix} z' & \overline{z'} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$ since Maass does not cover this derivation in his book.

We can factorise the first of these matrices as

$$\begin{pmatrix} z & \overline{z} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \overline{z} \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} z-\overline{z} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \qquad (3.34)$$

Since the first and last matrices in this factorisation are block triangular, they each have determinant equal to $\det(\mathbb{1})\det(\mathbb{1}) = 1$, and the middle matrix has determinant equal to $\det(z-\overline{z})\det(\mathbb{1}) = \det(z-\overline{z})$.

Thus,

$$\det\begin{pmatrix} z & \overline{z} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \det(z-\overline{z}) = 2i\cdot\det(y) \qquad (3.35)$$

and likewise,

$$\det\begin{pmatrix} z' & \overline{z'} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \det(z'-\overline{z'}) = 2i\cdot\det(y') \qquad (3.36)$$

We can now take determinants of both sides of (3.32) and obtain:

$$\begin{aligned}
\det(\gamma)\cdot 2i\cdot\det(y) &= 2i\cdot\det(y')\det(cz+d)\det(c\overline{z}+d) \\
\det(\gamma)\det(y) &= \det(y')\det(cz+d)\overline{\det(cz+d)} \\
\det(\gamma)\det(y) &= \det(y')|\det(cz+d)|^2 \qquad (3.37)
\end{aligned}$$

Since y and y' are both positive definite, $\det(y), \det(y') \geq 0$. Therefore we deduce that $\det(\gamma) \geq 0$.

Because we already had that $\det(\gamma) = \pm 1$ we can conclude that $\det(\gamma) = 1$ as required.

□

3.4.2 The group of Biholomorphisms of $\mathbb{H}_n$

It can be shown [17] that the group of Biholomorphisms of $\mathbb{H}_n$ is isomorphic to the symplectic group modulo plus or minus the identity:

$$\mathrm{Bihol}(\mathbb{H}_n) \cong \mathrm{Sp}(n,\mathbb{R})/\{\pm\mathbb{1}\} \qquad (3.38)$$

This is analagous to the classical case where the group of Biholomorphisms of the complex upper half-plane was isomorphic to the special linear group modulo plus or minus the identity

$$\mathrm{Bihol}(\mathbb{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm\mathbb{1}\} \qquad (3.39)$$

The proof is left to Appendix C since these results are not directly relevant to our discussion.

Chapter 4

Discrete Subgroups

Those familiar with classical modular forms will be aware of the idea of a ‘modular group’. In this chapter we shall explore how to generalise this idea to Siegel modular forms. Firstly we will consider some generalities surrounding discrete subgroups of the symplectic group.

4.1 Discrete Subgroups acting on $\mathbb{H}_n$

In order to define a modular group for $\mathrm{Sp}(n, \mathbb{R})$ we will first discuss discrete subgroups more generally, and define a number of subgroups of $\mathrm{Sp}(n, \mathbb{R})$ which will be useful later.

To provide context for the following we first recall the definition of an isolated point [27] from complex analysis:

Definition 15. *A point $w \in X \subset \mathbb{C}^d$ is called isolated if $\exists\ r > 0$ such that $B_r(w) \cap X = \{w\}$. A point which is not isolated is called an accumulation point.*

From here we follow Section 3 of Chapter 1 in Klingen’s book [17].

Definition 16. *Take G to be a subgroup of $\mathrm{Sp}(n, \mathbb{R})$. We say that G acts discontinuously on $\mathbb{H}_n$ if the group orbit*

$$\{\gamma\langle z\rangle : \gamma \in G\} \tag{4.1}$$

contains no accumulation point in $\mathbb{H}_n$ for any $z \in \mathbb{H}_n$.

Definition 17. *Take G to be a subgroup of $\mathrm{Sp}(n, \mathbb{R})$. We say that G is discrete if no sequence $\{g_v\}_{v\in\mathbb{N}}$ of mutually distinct elements $g_v \in G$ converges in G.*

Note that the above definition requires some notion of ‘distance’ in G in order to have some sense of ‘convergence’. We consider G as a subset of the space of $2n \times 2n$ matrices which is isomorphic as a metric space to $\mathbb{R}^{4n^2}$. So we consider a sequence to converge in G if the equivalent sequence of vectors in $\mathbb{R}^{4n^2}$ converges as per the usual definition.

We then have the following important result:

Proposition 18. *Let G be a subgroup of $\mathrm{Sp}(n, \mathbb{R})$. Then G acts discontinuously on $\mathbb{H}_n$ if and only if G is discrete.*

Klingen proves that discreteness $\implies$ discontinuity but leaves out the proof of discontinuity $\implies$ discreteness. We shall prove the latter for the sake of completeness:

Proof. We assume that G acts discontinuously on $\mathbb{H}_n$. So for all $z \in \mathbb{H}_n$ the set $\{\gamma\langle z\rangle : \gamma \in G\}$ has no accumulation point in $\mathbb{H}_n$.

Now we shall prove by contradiction that G must be discrete. Indeed assume that G is not discrete, so there is a convergent sequence $\{g_v\}_{v\in\mathbb{N}}$ with $g_v \in G$, such that $g_v \to g \in G$ as $v \to \infty$.

Without loss of generality we can assume that this sequence converges to $\mathbb{1}$. (We can see this by defining a new sequence as $g^{-1}g_v$.)

We now define a sequence in $\mathbb{H}_n$ by $z_v := g_v\langle z\rangle$ for some $z \in \mathbb{H}_n$. We want to show that this sequence converges to z.

Because the group action (for fixed $z \in \mathbb{H}_n$) is a continuous function from $\mathrm{Sp}(n,\mathbb{R}) \to \mathbb{H}_n$ it is clear to see that $g_v \to \mathbb{1}$ as $v \to \infty \implies g_v\langle z\rangle \to z$ as $v \to \infty$. (See Theorem 5.3 in [16])

Now we invoke the definition of convergence: $\forall\, r > 0, \exists\, N \in \mathbb{N}$ such that $\forall\, v > N, g_v\langle z\rangle \in B_r(z)$.

From this it is clear to see that $\nexists\, r > 0$ such that $\{\gamma\langle z\rangle : \gamma \in G\} \cap B_r(z) = \{z\}$. Thus the group orbit $\{\gamma\langle z\rangle : \gamma \in G\}$ contains an accumulation point, and so the group G does not act discontinuously on $\mathbb{H}_n$. Thus we arrive at a contradiction, since we assumed G acted discontinuously. Hence we conclude our assumption that G was not discrete was incorrect - indeed if G acts discontinuously then it must be discrete.

A proof of the converse implication can be found in Klingen's book [17] or alternatively in Appendix C. □

4.2 Siegel's Modular Group

We are now ready to introduce our most important example of a discrete group: The modular group.

Recall that for $n = 1$ we had the modular group $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ as a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. So it should be clear to see that in the case of general n we are concerned with the group $\mathrm{Sp}(n,\mathbb{Z})$ as a discrete subgroup of $\mathrm{Sp}(n,\mathbb{R})$ (and the subsequent induced action on $\mathbb{H}_n$).

Definition 19. *Siegel's modular group is defined as* $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$*, and elements* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$ *are called 'modular matrices' (as per Klingen [17].)*

Is this well defined as a group? It is fairly clear to see that Γ_n is closed under matrix multiplication - since all the entries will remain integer values if they are just obtained via adding and multiplying integers.

We should check that it is closed under inverses though. Maass [22] claims that $\gamma^{-1} = -j\,{}^t\gamma j$ for all $\gamma \in \mathrm{Sp}(n,\mathbb{R})$. We shall quickly check that this is indeed the case:

$$\gamma\gamma^{-1} = -\gamma j\,{}^t\gamma j = -j^2 = \mathbb{1} \tag{4.2}$$

where we used the condition $j[\gamma] = j$ used to define the symplectic group.

Now because j and ${}^t\gamma$ are both integral matrices for $\gamma \in \mathrm{Sp}(n,\mathbb{Z})$ it is clear to see that γ^{-1} is also an integral matrix, so indeed $\gamma^{-1} \in \Gamma_n$ and Siegel's modular group is well defined.

It is also clear to see that Γ_n has an induced action on $\mathbb{H}_n$ given by $\gamma\langle z\rangle = (az + b)(cz + d)^{-1}$.

4.3 Other discrete groups

We also have a number of other useful discrete groups which we can now define, given by Klingen [17] on page 28. Klingen's subsequent discussion of these subgroups is rather short and confusing so we will then continue with Maass [22] chapter 11 for a clearer view. The first of these is in fact not a subgroup of $\mathrm{Sp}(n,\mathbb{R})$ but rather a subgroup of $\mathrm{GL}(n,\mathbb{R})$.

4.3.1 The Unimodular Group

We now introduce the unimodular group [17], which will play an important role when finding the fundamental domain.

Definition 20. *The group* $U_n := \mathrm{GL}(n,\mathbb{Z})$ *is called the Unimodular group.*

Lemma 21. *For all* $u \in U_n$, $\det(u) = \pm 1$

Proof. Since the entries of u are all integers, we can deduce that $\det(u) \in \mathbb{Z}$. But $u^{-1} \in U_n$ as well, so we also require $\det(u^{-1}) = \det(u)^{-1} \in \mathbb{Z}$. The only way this is possible is if u has determinant ± 1. □

We denote by U_n^+ the subgroup of the unimodular group with positive determinant ($\mathrm{SL}_n(\mathbb{Z})$).

Whilst this group is not a subgroup of $\mathrm{Sp}(n,\mathbb{R})$, we can define a subgroup of $\mathrm{Sp}(n,\mathbb{R})$ which is isomorphic to U_n as follows:

$$\hat{U}_n := \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} : u \in U_n \right\} \cong U_n \tag{4.3}$$

It is fairly quick to see that this is indeed a subgroup of $\mathrm{Sp}(n,\mathbb{R})$ (and indeed $\mathrm{Sp}(n,\mathbb{Z})$). A proof can be found in Appendix C.

U_n induces a group action on $\mathbb{H}_n$. Via the definition of the group action of $\mathrm{Sp}(n,\mathbb{R})$ we get:

$$\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \langle z \rangle = (uz + 0)(0z + {}^t u^{-1})^{-1} = uz\, {}^t u = z[{}^t u] \tag{4.4}$$

4.3.2 The Translation Subgroup

Our next discrete subgroup is the group of translations of Siegel's half-space $\mathbb{H}_n$

$$T := \left\{ \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z}),\ {}^t s = s \right\} \tag{4.5}$$

Why is this a group of translations? Consider the action of the subgroup T on $\mathbb{H}_n$.

$$\begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \langle z \rangle = (\mathbb{1} z + s)(0z + \mathbb{1})^{-1} = z + s \tag{4.6}$$

So z is translated in the 'real directions' of $\mathbb{H}_n$ by some integral matrix s.

Remark 22. *When* $n = 1$ *the group* $T = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ *which is the group of real integral translations of the complex upper half plane* $\mathbb{H}$.

We should in theory prove that this is a subgroup of $\mathrm{Sp}(n,\mathbb{Z})$ but it is fairly trivial to work through the defining condition to check this.

4.3.3 Integral Modular Substitutions

Finally we will define a subgroup which Klingen [17] refers to as 'integral modular substitutions'.

Definition 23. *We define a subgroup A of* $\mathrm{Sp}(n,\mathbb{Z})$ *as*

$$A := \left\{ \begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} : u \in U_n;\ s \in \mathrm{M}_n(\mathbb{Z}),\ {}^t s = s \right\} \tag{4.7}$$

We will now prove a lemma and a proposition about how the group A can be constructed.

Lemma 24. $A = \{t\hat{u} : t \in T;\ \hat{u} \in \hat{U}_n\}$

Proof. It is clear to see how every matrix $\begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} \in A$ can be written as

$$\begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \tag{4.8}$$

where $t = \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \in T$ and $\hat{u} = \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} \in \hat{U}_n$ □

Proposition 25. $A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Z}) : c = 0 \right\}$

Klingen states this proposition as an throwaway comment at the end of page 28 [17] but it's a non-trivial fact which requires a proof:

Proof. We will first prove that every integral symplectic matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c = 0$ is in A.

We will make use of the symplecticity conditions. Firstly the condition ${}^t ad - {}^t cb = \mathbb{1}$ gives us

$${}^t ad = \mathbb{1} \tag{4.9}$$

when $c = 0$.

Since a and d have integer entries and are invertible by (4.9), they are in the unimodular group $\mathrm{GL}(n,\mathbb{Z})$. We set $u = a$ which means $d = {}^t u^{-1}$.

So we now have a matrix of the form $\begin{pmatrix} u & b \\ 0 & {}^t u^{-1} \end{pmatrix}$. We now want to show that b can be written as $s\,{}^t u^{-1}$ for some integral symmetric s. It therefore suffices to show that $b\,{}^t u$ is symmetric (clearly it has integer entries as both b and u do).

We first apply the symplecticity condition ${}^t bd = {}^t db$. Substituting in $d = {}^t u^{-1}$ gives us ${}^t b\,{}^t u^{-1} = u^{-1} b$ which rearranges to $u\,{}^t b = b\,{}^t u$.

We want to show that $b\,{}^t u$ is symmetric. Taking the transpose gives ${}^t(b\,{}^t u) = u\,{}^t b = b\,{}^t u$. So $b\,{}^t u$ is indeed symmetric and we can therefore write b as $s\,{}^t u^{-1}$ for some symmetric integral matrix s.

We have shown that all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Z})$ with $c = 0$ are in the group A. So indeed A is the group of all modular matrices with $c = 0$. □

4.3.4 Congruence Subgroups

We can define congruence subgroups of the modular group $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$ in an analogous way to the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. This won't be immediately useful but they will become relevant once we introduce Siegel modular forms and theta series in the later sections of this report.

Here we refer to the PhD thesis of Dickson [10] for the definition.

Definition 26. *The Hecke congruence subgroup of level N is given by:*

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) : c \equiv 0 \bmod N \right\} \tag{4.10}$$

Here the notation $c \equiv 0 \bmod N$ means that every entry in the matrix c is divisible by N

We will quickly prove a lemma which will be used in the final section of this report

Lemma 27. *When $N > 1$ we have that $\det(d) \neq 0$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(n)}(N)$.*

Proof. We will apply the symplecticity condition (2.19) modulo N. We get that ${}^t a d \equiv \mathbb{1} \bmod N$ since the term containing c is congruent to 0.

This allows us to take determinants modulo N to see that $\det({}^t a d) \equiv 1 \bmod N$, and thus $\det(a)\det(d) \equiv 1 \bmod N$.

If $\det(d) = 0$ then we would have $\det(d) \equiv 0 \bmod N$ which gives a contradiction. □

4.4 Associated Bottom Halves

We will now introduce a set of equivalence classes on modular matrices.

4.4.1 Bottom Halves of Modular Matrices

Definition 28. *The bottom half of a modular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix (c, d) which has n rows and $2n$ columns.*

Klingen [17] and Maass [22] both refer to the above as the 'second row' of γ but this language can be easily confused with the actual second row of entries of the matrix - hence our choice to use the term 'bottom half'.

We shall attempt to find some properties of modular matrices. In particular we are interested in finding conditions on the $n \times n$ blocks c and d which appear in the bottom half of modular matrices, and latterly we shall define important equivalence classes of bottom halves of modular matrices.

Klingen's discussion on page 28 is light on the explanation and difficult to follow. Maass [22], in chapter 11 of his book, gives some more detail on the logical steps to defining the equivalence classes but does not perform any calculations. We shall be more thorough in this section.

Definition 29. *Two matrices $c, d \in M_n(\mathbb{Z})$ are 'coprime' if whenever Pc and Pd are both matrices with integer entries, P is also a matrix with integer entries.*

Definition 30. *Two matrices $c, d \in M_n(\mathbb{Z})$ are a symmetric pair if $c\,{}^t d = d\,{}^t c$.*

Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$. Via the symplecticity conditions (2.19) we have that $c\,{}^t d = d\,{}^t c$ and also that $d\,{}^t a - c\,{}^t b = \mathbb{1}$. We will make use of these results in proving the following lemma

Lemma 31. *For all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$ *we have that* c *and* d *are a coprime symmetric pair.*

Maass [22] gives a sketch proof but leaves out the full multiplication and explanation. We shall complete it.

Proof. c and d are clearly a symmetric pair since we have $c\,{}^t d = d\,{}^t c$. We now show coprimality.

We take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$ and an arbitrary matrix $P \in M_{n\times n}(\mathbb{R})$ such that Pc and Pd are matrices with integer entries. Our goal is of course to show that these assumptions imply that P itself has integer entries.

Consider γ's bottom half (c, d) which has n rows and $2n$ columns, and multiply on the left by P:

$$P(c, d) = (Pc, Pd) \tag{4.11}$$

We have that Pc and Pd are integral matrices, so it follows that (Pc, Pd) must also be an integral matrix.

We now multiply on the right by γ^{-1}. This is a $2n \times 2n$ matrix in the group $\mathrm{Sp}(n, \mathbb{Z})$, so it has integer entries. We know the product of two integral matrices must also be an integral matrix, so $(Pc, Pd)\gamma^{-1}$ is an integral matrix (with n rows and $2n$ columns).

Let us multiply out this matrix - recalling that $\gamma^{-1} = -j\,{}^t\gamma j$:

$$\begin{aligned}
(Pc, Pd)\gamma^{-1} &= -(Pc, Pd)j\,{}^t\gamma j \\
&= (Pc, Pd)\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} {}^t a & {}^t c \\ {}^t b & {}^t d \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \\
&= (Pd, -Pc)\begin{pmatrix} -\,{}^t c & {}^t a \\ -\,{}^t d & {}^t b \end{pmatrix} \\
&= (-Pd\,{}^t c + Pc\,{}^t d, Pd\,{}^t a - Pc\,{}^t b) \\
&= (P(c\,{}^t d - d\,{}^t c), P(d\,{}^t a - c\,{}^t b))
\end{aligned} \tag{4.12}$$

We now apply the symplecticity conditions to get:

$$(Pc, Pd)\gamma^{-1} = (0, P) \tag{4.13}$$

We had deduced that $(Pc, Pd)\gamma^{-1}$ must be a matrix with integer entries, so $(0, P)$ must be a matrix with integer entries. We subsequently conclude that P is an integral matrix as required. □

Maass [22] on page 156 also proves the converse to this lemma: That all coprime symmetric pairs of matrices c and d form the bottom half of a modular matrix $\gamma \in \Gamma_n$.

4.4.2 A set of equivalence classes

Next we will introduce the notion of two bottom halves being 'associated' (as per the definition given by Maass [22]) which can also be seen as imposing a set of equivalence classes.

Definition 32. *Two bottom halves* (c_1, d_1) *and* (c_2, d_2) *are called 'associated' if* $c_1\,{}^t d_2 = d_1\,{}^t c_2$*.*

We denote the set of all bottom halves (c', d') *associated to* (c, d) *by* $[c, d]$*.*

Similarly, we denote the set of all $\gamma' \in \Gamma_n$ *such that the bottom half of* γ' *is associated to the bottom half of* γ *by* $[\gamma]$*.*

We will now prove a series of results which will help us classify these equivalence classes.
Maass [22] states all of these results but the proofs are left out.
First we will consider right cosets of T.

Proposition 33. *Take* $\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ *and* $\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$.
Then the cosets $T\gamma_1$ *and* $T\gamma_2$ *are equal if and only if* $c_1 = c_2$ *and* $d_1 = d_2$.

Proof. Writing out the cosets in full gives:

$$\begin{aligned} T\gamma_1 &= \left\{ \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z});\ {}^t s = s \right\} \\ T\gamma_2 &= \left\{ \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z});\ {}^t s = s \right\} \end{aligned} \tag{4.14}$$

Then multiplying out the matrices gives us a useful characterisation of our two cosets:

$$\begin{aligned} T\gamma_1 &= \left\{ \begin{pmatrix} a_1 + sc_1 & b_1 + sd_1 \\ c_1 & d_1 \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z});\ {}^t s = s \right\} \\ T\gamma_2 &= \left\{ \begin{pmatrix} a_2 + sc_2 & b_2 + sd_2 \\ c_2 & d_2 \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z});\ {}^t s = s \right\} \end{aligned} \tag{4.15}$$

We shall first assume that the cosets are equal. We see from the above characterisation that every matrix in $T\gamma_1$ has bottom half equal to (c_1, d_1) and every matrix in $T\gamma_2$ has bottom half equal to (c_2, d_2). So if $T\gamma_1 = T\gamma_2$ it must mean that $(c_1, d_1) = (c_2, d_2)$.

Conversely we now assume that $(c_1, d_1) = (c_2, d_2) =: (c, d)$ and we endeavour to show that $T\gamma_1 = T\gamma_2$. This is equivalent to showing $T\gamma_1\gamma_2^{-1} = T$ and thus it suffices to show that $\gamma_1\gamma_2^{-1} \in T$.

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix} \begin{pmatrix} {}^t d & -{}^t b_2 \\ -{}^t c & {}^t a_2 \end{pmatrix} = \begin{pmatrix} a_1\,{}^t d - b_1\,{}^t c & -a_1\,{}^t b_2 + b_1\,{}^t a_2 \\ c\,{}^t d - d\,{}^t c & -c\,{}^t b_2 + d\,{}^t a_2 \end{pmatrix} \tag{4.16}$$

By symplecticity of γ_1 and γ_2 we can reduce this to

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} \tag{4.17}$$

where we define $s := -a_1\,{}^t b_2 + b_1\,{}^t a_2$. In order to show $\gamma_1\gamma_2^{-1} \in T$ we need to show that s has integer entries and is symmetric. It is clearly integral due to a_1, b_1, a_2 and b_2 all being integral.

To show symmetry is more tricky. Note that because γ_1 and γ_2 are both in $\mathrm{Sp}(n, \mathbb{R})$, then so is $\gamma_1\gamma_2^{-1}$. So this means we can apply the symplecticity conditions on $\gamma_1\gamma_2^{-1}$.

We see via the condition $a\,{}^t b = b\,{}^t a$ with $a = \mathbb{1}$ and $b = s$ that $\mathbb{1}\,{}^t s = s\,{}^t \mathbb{1}$. Hence ${}^t s = s$ and so $\gamma_1\gamma_2^{-1} \in T$ and we are done. □

Lemma 34. *Two matrices* $\gamma_1, \gamma_2 \in \Gamma_n$ *have associated bottom halves if and only if* $A\gamma_1 = A\gamma_2$ *as cosets.*

Maass [22] states this result whilst omitting the matrix algebra so we shall make sure to fully justify this statement.

Proof. We shall begin by assuming that $[\gamma_1] = [\gamma_2]$. To show $A\gamma_1 = A\gamma_2$ it suffices to prove that $\gamma_1\gamma_2^{-1} \in A$.

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} {}^t d_2 & -{}^t b_2 \\ -{}^t c_2 & {}^t a_2 \end{pmatrix} = \begin{pmatrix} * & * \\ c_1\,{}^t d_2 - d_1\,{}^t c_2 & * \end{pmatrix} \tag{4.18}$$

We know that the entries marked with $*$s are integral matrices so it's not important to calculate their value. We only need to show that the bottom left block is equal to zero.

By our assumption that the bottom halves were associated we see that $c_1\,{}^t d_2 - d_1\,{}^t c_2 = 0$. Therefore we have $\gamma_1\gamma_2^{-1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in A$.

Now we assume that $A\gamma_1 = A\gamma_2$. In other words we assume that $\gamma_1\gamma_2^{-1} \in A$. By the above 4.18 we see that we require $c_1\,{}^t d_2 - d_1\,{}^t c_2 = 0$ for this to be the case. Hence $c_1\,{}^t d_2 = d_1\,{}^t c_2$ and the matrices γ_1 and γ_2 have associated bottom halves.

□

We will now prove another result about associated bottom halves.

Proposition 35. *Two bottom halves (c_1, d_1) and (c_2, d_2) are associated if and only if there exists a $u \in U_n$ such that $(c_1, d_1) = u(c_2, d_2)$.*

Klingen [17] introduces this as the definition of bottom halves being associated, whilst Maass [22] states this as a fact which follows immediately from the above lemma. We shall prove that this proposition follows from Maass' characterisation because this will convince us that Klingen and Maass' definitions do indeed describe the same concept.

Proof. Consider $\gamma_1, \gamma_2 \in \Gamma_n$ with bottom halves (c_1, d_1) and (c_2, d_2) respectively.

First assume that $(c_1, d_1) = u(c_2, d_2)$ for some $u \in U_n$. We want to show that the two bottom halves are associated, which equates to showing that $c_1\,{}^t d_2 = d_1\,{}^t c_2$. The right hand side can be written as

$$d_1\,{}^t c_2 = u d_2\,{}^t c_2 \tag{4.19}$$

and we can then use symplecticity of γ_2 to get

$$d_1\,{}^t c_2 = u c_2\,{}^t d_2 = c_1\,{}^t d_2 \tag{4.20}$$

So we have shown the first implication. The converse is a little less obvious.

We now assume that $[\gamma_1] = [\gamma_2]$. Then it follows from Lemma 34 that $A\gamma_1 = A\gamma_2$. So therefore it must be true that $\gamma_1 \in A\gamma_2$, and hence there exists some matrix $P = \begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} \in A$ such that $\gamma_1 = P\gamma_2$ (with s symmetric and u unimodular).

Thus we have

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} u a_2 + s\,{}^t u^{-1} c_2 & u b_2 + s\,{}^t u^{-1} d_2 \\ {}^t u^{-1} c_2 & {}^t u^{-1} d_2 \end{pmatrix} \tag{4.21}$$

So we have that $c_1 = {}^t u^{-1} c_2$ and $d_1 = {}^t u^{-1} d_2$ and therefore $(c_1, d_1) = {}^t u^{-1}(c_2, d_2)$. Since u is unimodular so is ${}^t u^{-1}$ and thus we are done.

□

We now have some important characterisations of modular matrices with associated bottom halves. This will come back to help us later in the next section.

Chapter 5

The Fundamental Domain

According to Maass [22], we define a fundamental set as follows:

Definition 36. *A fundamental set (or domain) for a group* Γ *acting on a set* P *is a subset of* P *consisting of a single representative element from each equivalence class (or group orbit)* $\mathcal{O}(p) = \{\gamma\langle p\rangle : \gamma \in \Gamma\}$ *which is in some way 'reduced'.*

Our ultimate goal will be to find a fundamental set for $\mathrm{Sp}(n, \mathbb{Z})$ acting on $\mathbb{H}_n$. The first step in this process will actually be to find a fundamental set for the unimodular group U_n acting on the imaginary part of $\mathbb{H}_n$ which we denoted by $\mathbb{H}_n^{\mathrm{Im}}$ in (2.4). We call this set 'Minkowski's reduced domain'

5.1 Minkowski's reduced domain

We will refer to the discussion in section 2 of Klingen's book [17]. We will be brief in our study of Minkowski's reduced domain as to fully understand it would be beyond the scope of this report.

We recall from (4.4) that the unimodular group acts on $\mathbb{H}_n$ by $z[{}^t u]$ for $z \in \mathbb{H}_n$ and $u \in U_n$. By letting $z = x + iy$ for $x \in \mathbb{H}_n^{\mathrm{Re}}$ and $y \in \mathbb{H}_n^{\mathrm{Im}}$, we see that this induces an action of U_n on $\mathbb{H}_n^{\mathrm{Im}}$ by:

$$\begin{array}{rcl} U_n \times \mathbb{H}_n^{\mathrm{Im}} & \to & \mathbb{H}_n^{\mathrm{Im}} \\ (u, y) & \mapsto & y[{}^t u] \end{array} \tag{5.1}$$

We will construct 'Minkowski's reduced domain' to be a fundamental set for the action of U_n on $\mathbb{H}_n^{\mathrm{Im}}$ defined in this way.

Definition 37. *Minkowski's reduced domain is the set*

$$R_n = \{y \in \mathbb{H}_n^{\mathrm{Im}} : y \text{ satisfies } (i) \text{ and } (ii) \text{ below}\} \tag{5.2}$$

with

(i) For each k *with* $1 \leq k \leq n$*:*
$y[\mathbf{g}] \geq y_{k,k}$ *for all* $\mathbf{g} \in \mathbb{Z}^n$ *such that* $\gcd(g_k, g_{k+1}, ..., g_n) = 1$.

(ii) For each k *with* $1 \leq k \leq n-1$*:*
$y_{k,k+1} \geq 0$

Klingen shows that this is a fundamental domain for the action of U_n on $\mathbb{H}_n^{\mathrm{Im}}$. [17]

5.2 Siegel's Fundamental Domain

We can now discuss the full fundamental domain for the action of $\mathrm{Sp}(n,\mathbb{Z})$ on $\mathbb{H}_n$.

Recall that in the case of classical modular forms [12] we are able to define a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$. Indeed it can be shown that the set

$$\mathcal{F} = \{\tau \in \mathbb{H} : |\tau| \geq 1 \text{ and } |\mathrm{Re}(\tau)| \leq \frac{1}{2}\} \tag{5.3}$$

contains a representative of every group orbit, and that the points in the interior of $\mathcal{F}$ are in different group orbits.

In this subsection we will define an analogue of $\mathcal{F}$ for Siegel modular forms called 'Siegel's fundamental domain' and show that it contains at least one point in every group orbit.

Definition 38. *Given any symplectic map $z \mapsto \gamma\langle z\rangle$, the factor of automorphy [17] is*

$$j(\gamma, z) := \det(cz + d) \tag{5.4}$$

Lemma 39. *The factor of automorphy satisfies the cocycle relation:*

$$\begin{aligned} j(\gamma_1\gamma_2, z) &= j(\gamma_1, \gamma_2\langle z\rangle)j(\gamma_2, z) \\ &\forall \quad z \in \mathbb{H}_n, \gamma_1, \gamma_2 \in \mathrm{Sp}(n,\mathbb{R}) \end{aligned} \tag{5.5}$$

Proof. Let $\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$\begin{aligned} j(\gamma_1, \gamma_2\langle z\rangle)j(\gamma_2, z) &= \\ \det(c_1\gamma_2\langle z\rangle + d_1)\det(c_2 z + d_2) &= \det(c_1(a_2 z + b_2)(c_2 z + d_2)^{-1} + d_1)\det(c_2 z + d_2) \\ = \det(c_1(a_2 z + b_2) + d_1(c_2 z + d_2)) &= \det((c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2) \end{aligned} \tag{5.6}$$

Now we note that $\gamma_1\gamma_2 = \begin{pmatrix} * & * \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$ and thus
$j(\gamma_1\gamma_2, z) = \det((c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2)$ and so we have the cocycle relation
$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2\langle z\rangle)j(\gamma_2, z)$ □

This concept will be more relevant once we define Siegel modular forms, but it is convenient to introduce it now to allow us to prove the following lemma:

Lemma 40. *Take a fixed $z \in \mathbb{H}_n$ and $\gamma_1, \gamma_2 \in \Gamma_n$ two modular matrices such that $[\gamma_1] = [\gamma_2]$. Then we have:*

$$|j(\gamma_1, z)| = |j(\gamma_2, z)| \tag{5.7}$$

Proof. Let the bottom halves of γ_1 and γ_2 be (c_1, d_1) and (c_2, d_2) respectively. By Proposition 35, there exists a $u \in U_n$ such that $c_1 = uc_2$ and $d_1 = ud_2$. Therefore, we can deduce

$$|j(\gamma_1, z)| = |\det(c_1 z + d_1)| = |\det(uc_2 z + ud_2)| = |\det(u(c_2 z + d_2))| \tag{5.8}$$

$$= |\det(u)\det(c_2 z + d_2)| = |\det(c_2 z + d_2)| = |j(\gamma_2, z)| \tag{5.9}$$

where we make the final step due to the fact that $\det(u) = \pm 1$ for all u in the unimodular group. □

So this lemma states that the absolute value of the automorphy factor is constant on equivalence classes of matrices with associated bottom halves. We now state a powerful proposition about the number of equivalence classes with automorphy factor of size less than a given bound.

5.2.1 A result on equivalence classes

Proposition 41. *For a fixed $z \in \mathbb{H}$, and real number $\lambda > 0$, there exists only finitely many equivalence classes $[\gamma]$ of modular matrices with associated bottom halves such that*

$$|j(\gamma, z)| < \lambda \tag{5.10}$$

Klingen [17] states the above as a lemma, and gives a sketch proof. There is a particular step which requires a significant amount of algebraic manipulation to prove which is omitted. We shall detail the full proof whilst following the original structure.

In order to enact Klingen's proof strategy we will need the help of a lemma stated on page 20 of his book [17]

Lemma 42. *Let $y \in R_n$ be a point in Minkowski's reduced domain, with diagonal elements given by y_i for $i = 1, ..., n$.*

Denote by y^D the matrix constructed out of just the diagonal elements of y:

$$y^D := \mathrm{diag}(y_1, y_2, ..., y_n) \tag{5.11}$$

Then for all $y \in R_n$ there exists a constant $C > 0$ depending only on n such that

$$C^{-1}y < y^D < Cy \tag{5.12}$$

where here the notation $A > B$ for matrices A and B denotes the Loewner order - that $A - B$ is positive definite. [4]

Confusingly, Klingen refers to y^D as the 'diagonalization of y' but this does not coincide with the notion of the same name from Linear Algebra - so we will avoid this language.

We will not give a proof here for the sake of brevity but Klingen's book goes into more detail.

We are now able to provide a proof of proposition 41.

Proof. Let $z = x + iy \in \mathbb{H}_n, \lambda \in \mathbb{R}_{>0}$ and pick $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$ such that $|\det(cz + d)| < \lambda$. Lastly denote $z' = x' + iy' = \gamma\langle z\rangle$.

Recall the transformation formula for the imaginary part y under the group action (3.25):

$$y' = y\{\overline{(cz + d)}^{-1}\} \tag{5.13}$$

We can rearrange this to get a similar formula for y'^{-1}:

$$\begin{aligned} y' &= y\{\overline{(cz+d)}^{-1}\} \\ \implies y' &= {}^t(cz+d)^{-1} y \overline{(cz+d)}^{-1} \\ \implies {}^t(cz+d)y' &= y\overline{(cz+d)}^{-1} \\ \implies y^{-1}\, {}^t(cz+d) &= \overline{(cz+d)}^{-1} y'^{-1} \\ \implies y'^{-1} &= \overline{(cz+d)} y^{-1}\, {}^t(cz+d) \\ \implies y'^{-1} &= y^{-1}\{{}^t(cz+d)\} \end{aligned} \tag{5.14}$$

We now multiply c and d by a $u \in U_n$ such that y'^{-1} is reduced in the sense of Minkowski (Definition 37). We can do this because the group action of U_n on $\mathbb{H}_n^{\mathrm{Im}}$ was induced by the action of $\hat{U}_n$ on $\mathbb{H}_n$ - and multiplying γ by $\hat{u} \in \hat{U}_n$ has the effect of multiplying c and d by $u \in U_n$.

Essentially this boils down to picking a new representative of the equivalence class $[\gamma]$ such that $y'^{-1} \in R_n$.

We now denote the diagonal elements of y'^{-1} by μ_k and the rows of c and d by c_k and d_k for $k \in \{1, 2, 3, ..., n\}$.

We now have to do some heavy algebraic manipulation. Klingen [17] misses this out as it is rather long but it will be rewarding to see how the formula he states can be derived.

First note that by (5.14), the i,jth entry of y'^{-1} is given by

$$[y'^{-1}]_{ij} = \overline{(cz+d)_i} y^{-1}\, {}^t(cz+d)_j \tag{5.15}$$

Thus $\mu_k = \overline{(cz+d)_k} y^{-1}\, {}^t(cz+d)_k = \overline{(c_k z + d_k)} y^{-1}\, {}^t(c_k z + d_k)$.

We now rearrange this equation:

$$\begin{aligned} \mu_k &= \overline{(c_k z + d_k)} y^{-1}\, {}^t(c_k z + d_k) \\ &= (c_k \overline{z} + d_k) y^{-1}\, {}^t(c_k z + d_k) \\ &= (c_k(x - iy) + d_k) y^{-1}\, {}^t(c_k(x+iy) + d_k) \\ &= (c_k x + d_k - i c_k y) y^{-1}\, {}^t(c_k x + d_k + i c_k y) \end{aligned} \tag{5.16}$$

Expanding out the first bracket as real and imaginary parts:

$$\begin{aligned} &= (c_k x + d_k) y^{-1}({}^t(c_k x + d_k) + i\, {}^t(c_k y)) \\ &\quad - i c_k y y^{-1}({}^t(c_k x + d_k) + i\, {}^t(c_k y)) \end{aligned} \tag{5.17}$$

And now doing the same with the second bracket:

$$\begin{aligned} &= y^{-1}[{}^t(c_k x + d_k)] + (c_k x + d_k) y^{-1} i\, {}^t(c_k y) \\ &\quad - i c_k\, {}^t(c_k x + d_k) + c_k\, {}^t(c_k y) \end{aligned}$$

We now expand out the ${}^t(c_k y)$ terms whilst recalling that y is symmetric:

$$\begin{aligned} &= y^{-1}[{}^t(c_k x + d_k)] + i(c_k x + d_k) y^{-1} y\, {}^t c_k \\ &\quad - i c_k\, {}^t(c_k x + d_k) + c_k y\, {}^t c_k \\ &= y^{-1}[{}^t(c_k x + d_k)] + i(c_k x + d_k)\, {}^t c_k \\ &\quad - i c_k\, {}^t(c_k x + d_k) + y[{}^t c_k] \end{aligned} \tag{5.18}$$

We now focus on the two middle terms. Notice that $i(c_k x + d_k)\, {}^t c_k = {}^t(i c_k\, {}^t(c_k x + d_k))$. Also because the c_k and d_k are row vectors, these quantities are just scalars. This means that ${}^t(i c_k\, {}^t(c_k x + d_k)) = i c_k\, {}^t(c_k x + d_k)$ and thus the middle terms just cancel with eachother.

We are therefore left with:

$$\mu_k = y^{-1}[{}^t(c_k x + d_k)] + y[{}^t c_k] \tag{5.19}$$

which is the formula given by Klingen [17].

Since y is positive definite (and thus y^{-1} as well), and the c_k,d_k are integral vectors not both equal to $\mathbf{0}$, we see via (5.19) that each μ_k must be bounded below by some positive constant independent of c and d.

Now we can deduce from (5.14) that

$$\det(y'^{-1}) = \overline{\det(cz+d)}\det(y^{-1})\det(cz+d) \tag{5.20}$$

and thus via our assumption that $|\det(cz+d)| < \lambda$ we get:

$$\det(y'^{-1}) \leq \lambda^2 \det(y^{-1}) \tag{5.21}$$

Via Lemma 42 (which was proven on page 20 of Klingen's book [17]) we have $y'^{-1} > C\mu$ for some positive constant C, where here $\mu = \mathrm{diag}(\mu_1, ..., \mu_n)$

It is a known result (details in Appendix C) that if $A, B, A-B$ are all positive definite, then $\det(A) > \det(B)$.

Hence in the inequality (5.21) we can replace y'^{-1} by the matrix μ up to an overall constant. Thus we have

$$\det(\mu) = \prod_k \mu_k \leq \lambda^2 \det(y^{-1}) \tag{5.22}$$

So we know that the product of the μ_k are bounded above by a constant independent of c and d. It immediately follows that each individual μ_k is bounded above by a positive constant independent of c and d. We had from before that each μ_k was in fact bounded below by some positive constant independent of c and d.

Therefore each μ_k is bounded above and below by constants independent of our choice of γ. By (5.19) we can therefore see that there are only a finite number of choices for c_k and d_k in order for those bounds to hold. Thus there are only a finite number of equivalence classes of associated bottom halves of modular matrices $\gamma \in \Gamma_n$ such that $|\det(cz+d)| < \lambda$. □

5.2.2 Heights of points

With a few extra deductions we will be able to use the above proposition to define a fundamental domain for Γ_n. Klingen [17] states the following facts but glosses over the logical connections.

Definition 43. *For $z = x + iy \in \mathbb{H}_n$ we call* $\det(y)$ *the 'height' of that point.*

Remark 44. *When $n = 1$ this is analogous to the size of the imaginary part of the number $z = x+iy$, which in the complex upper half plane $\mathbb{H}$ is the 'height' above the real axis.*

Lemma 45. *Each group orbit of Γ_n contains points of maximal height.*

Proof. For $z = x + iy \in \mathbb{H}_n$ we shall denote the group orbit of z by $\mathcal{O}(z)$.

Take any $z \in \mathbb{H}_n$ and we act on it by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z})$ such that $z' = x' + iy' = \gamma\langle z \rangle$. By the transformation formula for the imaginary part (3.25) we have $y' = y\{\overline{(cz+d)}^{-1}\}$. Taking determinants on both sides gives us the formula:

$$|\det(cz+d)|^2 \det(y') = \det(y) \tag{5.23}$$

Now we know by Proposition 41 that for any $z \in \mathbb{H}_n$ there are only finitely many equivalence classes $[\gamma]$ such that $|j(\gamma, z)| = |\det(cz+d)| < 1$ (By setting $\lambda = 1$).

Hence plugging this bound into (5.23) implies that there are finitely many equivalence classes $[\gamma]$ such that

$$\det(y) < \det(y') \tag{5.24}$$

We now proceed via contradiction. Assume that in $\mathcal{O}(z)$ that there do not exist points of maximal height. We pick a point z with height $\det(y)$. We want to find a point in the group orbit with a larger height.

Recall by Lemma 40 that the automorphy factor $|j(\gamma, z)| = |\det(cz+d)|$ is the same within each equivalence class. So we pick an equivalence class and a representative of that class γ such that the height of $z' = \gamma\langle z\rangle$ is equal to $\det(y') > \det(y)$.

We now repeat this process with z' to find another point with greater height. However this process must eventually terminate as there are only finitely many equivalence classes of modular matrices with associated bottom halves such that the inequality (5.24) holds.

Therefore the point at which this process terminates is a point of maximal height in $\mathcal{O}(z)$ □

Lemma 46. *Take a group orbit $\mathcal{O}(z)$ of Γ_n and assume without loss of generality that z is a point in this orbit with maximal height. Then we have:*

$$|\det(cz+d)| \geq 1 \tag{5.25}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n$.

Proof. Let $z' = x' + iy' = \gamma\langle z\rangle$ and recall that we have the formula $|\det(cz+d)|^2 \det(y') = \det(y)$. If z was a point of maximal height then we have $\det(y) \geq \det(y')$ for all $\gamma \in \Gamma_n$.

Plugging the inequality into the determinant transformation formula gives $|\det(cz+d)|^2 \geq 1$ for all $\gamma \in \Gamma_n$ and the result follows immediately. □

There are, in fact, an infinite number of points of maximal height in each group orbit. This comes from the below lemma:

Lemma 47. *Take $\gamma \in A$ and $z \in \mathbb{H}_n$. The height of $\gamma\langle z\rangle$ is equal to the height of z*

Proof. Take an arbitrary $\gamma = \begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix}$ where $u \in U_n$ and s is an integral symmetric matrix.

The quantity $|\det(cz+d)| = |0z + {}^t u^{-1}| = 1$ so by (5.23) we have $\det(y') = \det(y)$ and thus the height remains unchanged under the action of the group A. □

5.2.3 Defining the Fundamental Domain

These results motivate the definition of Siegel's Fundamental Domain:

Definition 48. *Let $n \in \mathbb{N}$. Siegel's Fundamental domain is a subset $\mathcal{F}_n \subset \mathbb{H}_n$ where elements $z = x + iy \in \mathcal{F}_n$ satisfy the following conditions:*

1. $|j(\gamma, z)| \geq 1$ *for all* $\gamma \in \mathrm{Sp}(n, \mathbb{Z})$
2. $y \in R_n$ *where* R_n *is Minkowski's reduced domain*
3. $|x_{ij}| \leq 1/2$ *for all* $1 \leq i \leq j \leq n$

Theorem 49. *Each group orbit of* Γ_n *contains at least one point in* $\mathcal{F}_n$.

Klingen [17] briefly mentions the strategy for completing the proof so we shall fill out the logical steps.

Proof. Take a group orbit $\mathcal{O}(z)$. We will transform z by matrices in Γ_n such that the image is in $\mathcal{F}_n$.

By Lemma 45 we can transform z by some $\gamma_1 \in \Gamma_n$ such that $\gamma_1\langle z\rangle$ has maximal height. By Lemma 46 this point has the property that $|j(\gamma, z)| \geq 1$ for all $\gamma \in \Gamma_n$. Thus we have found a point in $\mathcal{O}(z)$ satisfying the first criterion for being in $\mathcal{F}_n$.

We now restrict ourselves to acting with matrices in the subgroup A. By Lemma 47 these matrices do not effect the height - and thus acting on our point $\gamma_1\langle z\rangle$ will take it to other points of maximal height which will also satisfy the first criterion.

We now recall that the groups $\hat{U}_n$ and T are subgroups in A. We recall their respective induced group actions take $z \mapsto z[{}^t u]$ and $z \mapsto z + s$ for some $u \in U_n$ and real integral symmetric s.

First we act with $\gamma_2 \in \hat{U}_n$ such that the imaginary part of $\gamma_2\gamma_1\langle z\rangle$ is in Minkowski's reduced domain. This means we now have a point in our group orbit satisfying the first two criteria for $\mathcal{F}_n$.

Finally, we note that acting with a matrix $\gamma_3 \in T$ corresponding to the translation $\gamma_2\gamma_1\langle z\rangle \mapsto \gamma_2\gamma_1\langle z\rangle + s$ only effects the real part of $\gamma_2\gamma_1\langle z\rangle$ - thus it won't change the fact that the imaginary part is in R_n.

Therefore we pick an s such that the entries $s_{ij} = -[x'_{ij}]$ (where x' is the real part of $\gamma_2\gamma_1\langle z\rangle$, and $[\alpha]$ denotes α rounded to the nearest integer)

We act with $\gamma_3 = \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix}$ and we get a point $\gamma_3\gamma_2\gamma_1\langle z\rangle =: \hat{z} = \hat{x} + i\hat{y}$. The entries of the real part are calculated to be $\hat{x}_{ij} = x'_{ij} - [x'_{ij}]$. Since $[x'_{ij}]$ is just x'_{ij} rounded to the nearest integer, we get that the difference $-1/2 \leq x'_{ij} - [x'_{ij}] \leq 1/2$.

Thus we have a point $\hat{z}$ with $|\hat{x}_{ij}| \leq 1/2$ for all $1 \leq i \leq j \leq n$.

$\hat{z}$ thus satisfies the three criteria necessary for $\hat{z} \in \mathcal{F}_n$. So we have proven that each group orbit $\mathcal{O}(z)$ contains a point in Siegel's Fundamental Domain. □

In order for $\mathcal{F}_n$ to be a 'true' fundamental domain we would need to show that each group orbit only has one representative in the domain - however we will now move on and cover some more informative topics related to Siegel modular forms. Klingen [17] and Maass [22] both cover the topic of the Fundamental domain in more detail.

Chapter 6

Siegel Modular Forms

We now have all the tools required to define Siegel modular forms. We shall begin by defining Siegel modular forms for the full modular group $\mathrm{Sp}(n, \mathbb{Z})$ as per the book of Klingen [17]. We will give an overview of Fourier series for Siegel modular forms which will follow the notes of Kohnen [19] as well as drawing from Klingen [17]. Then we will define modular forms more generally for Hecke subgroups $\Gamma_0^{(n)}(N)$ as per the PhD thesis of Dickson [10], which will allow us to discuss theta series.

6.1 Siegel Modular Forms for $\mathrm{Sp}(n, \mathbb{Z})$

Definition 50. *A function $f : \mathbb{H}_n \to \mathbb{C}$ is called a Siegel modular form for $\mathrm{Sp}(n, \mathbb{Z})$ of degree n and weight k if*

(i) f is holomorphic

(ii) $f(\gamma\langle z\rangle) = j(\gamma, z)^k f(z)$ for all $\gamma \in \mathrm{Sp}(n, \mathbb{Z})$ and $z \in \mathbb{H}_n$.

(iii) f is bounded on Siegel's Fundamental Domain.

It can be shown, as by Klingen [17] on page 45 of his book, that condition *(iii)* actually follows as a corollary of *(i)* and *(ii)* when $n > 1$ by the 'Koecher principle'. This means that if we are given a function which we assume is not classical (i.e. $n \geq 2$), then it is much easier to check whether it is a modular form compared to the classical case. We will discuss the Koecher principle after briefly introducing Fourier Series.

Remark 51. *It can be shown that the Siegel modular forms of degree n and weight k form a vector space over $\mathbb{C}$.*

6.1.1 Fourier Series

We recall that in the case of classical modular forms we may write a modular form as a Fourier series (or q-series) [12]. It is possible to generalise Fourier series to functions of multiple complex variables, and in particular functions on $\mathbb{H}_n$.

We will not endeavour to introduce Fourier Theory rigorously for functions on $\mathbb{H}_n$ as to not get distracted from the main subject but we will give a short discussion which will be relevant to Siegel modular forms. This discussion is mostly based on the notes of Kohnen [19] as well as the book of Klingen [17].

Lemma 52. *For* $\gamma = \begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} \in A$ *with* s *symmetric and* $u \in U_n$ *unimodular, and* f *a modular form of weight* k *and degree* n*, we have:*

$$f(\gamma\langle z\rangle) = \det(u)^k f(z) \tag{6.1}$$

Moreover, it follows that when nk *is odd, the only Siegel modular form of weight* k *and degree* n *is the zero function* $f(z) \equiv 0$

Proof. We recall from Lemma 24 that $\gamma = t\hat{u}$ where $t = \begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix}$ and $\hat{u} = \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix}$. Via the cocycle relation we have $j(t\hat{u}, z) = j(t, \hat{u}\langle z\rangle)j(\hat{u}, z)$. We can calculate the factor of automorphy for t and $\hat{u}$ acting on $z \in \mathbb{H}_n$:

$$\begin{aligned} j(t, z) &= \det(0z + \mathbb{1}) = 1 \\ j(\hat{u}, z) &= \det(0z + {}^t u^{-1}) = \det(u) \end{aligned} \tag{6.2}$$

Plugging these into the cocycle relation gives $j(\gamma, z) = \det(u)$. Then substituting this into the transformation law for Siegel modular forms gives the required result.

We now take nk to be odd. This means both n and k are odd.

Take $\gamma = -\mathbb{1}_{2n\times 2n}$. So $s = 0$ and $u = -\mathbb{1}_{n\times n}$. We have that $-\mathbb{1}_{2n\times 2n}\langle z\rangle = z$. We also have that $\det(u) = \det(-\mathbb{1}_{n\times n}) = -1$. Plugging this into the transformation law we just proved gives

$$f(z) = (-1)^k f(z) = -f(z) \tag{6.3}$$

and thus $f(z) \equiv 0$.

□

Proposition 53. *Let* f *be a Siegel modular form of weight* k *and degree* n*. Then* f *has a 'Fourier Expansion' which is absolutely convergent for* $z \in \mathbb{H}_n$*:*

$$f(z) = \sum_G a(G)\exp(\pi i \sigma(Gz)) \tag{6.4}$$

where the sum runs over all symmetric positive semi-definite $G \in \mathrm{M}_n(\mathbb{Z})$ *with even entries on the diagonal,* $\sigma(m)$ *denotes the trace of a matrix* m*, and* $a(G)$ *are some coefficients depending on* G*.*

Equivalently we can also write

$$f(z) = \sum_H a(H)\exp(2\pi i \sigma(Hz)) \tag{6.5}$$

where here we have set $H = \frac{1}{2}G$ *and we now let* H *run over all matrices with half-integral entries and integers on the diagonal.*

For the sake of time we shall not prove the convergence of these series in general. Kohnen's notes [19], Klingen's book [17] and Van Der Geer's notes [13] all claim these series to converge. Instead we shall just briefly outline why one gets the expansion (6.5).

Proof. We recall that under the action of the translation subgroup T, we have that $t\langle z\rangle = z + s$ where s is a symmetric integer valued matrix.

By Lemma 52 we have that $f(z + s) = f(z)$ for all $z \in \mathbb{H}_n$ and for all integral symmetric s. Thus, as per Klingen [17], f is periodic in all variables z_{ij} $(i \leq j)$ with period 1.

The trace of Hz is given by $\sigma(Hz) = \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}z_{ji}$

Thus we can deduce that

$$\begin{aligned} \sigma(H(z+s)) &= \sigma(Hz) + \sigma(Hs) = \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}z_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}s_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}z_{ji} + 2\sum_{1\leq i<j\leq n} H_{ij}s_{ij} + \sum_{i=1}^{n} H_{ii}s_{ii} \\ &= \sigma(Hz) + N \end{aligned} \tag{6.6}$$

where here N is some integer since $2H_{ij}$ is an integer for $i \neq j$ and H_{ii} is an integer for all i. Thus we have

$$\exp(2\pi i\sigma(H(z+s))) = \exp(2\pi i\sigma(Hz) + 2\pi iN) = \exp(2\pi i\sigma(Hz)) \tag{6.7}$$

So the series expansion (6.5) transforms in the correct way under the action of the translation subgroup T. Combining this with the fact that f is holomorphic allows us to see that f can indeed be written as a Fourier series. □

Lemma 54. *For all H in the Fourier expansion (6.5) and $u \in U_n$ we have*

$$a(H[u]) = \det(u)^k a(H) \tag{6.8}$$

This lemma is stated by Kohnen [19] without proof, and by Klingen [17] with a proof that references the intgeral formula for the Fourier coefficients. We will present an alternate proof which does not require a formula for calculating the coefficients.

Proof. Applying Lemma 52 with $s = 0$ and recalling that $\hat{U}_n$ acts on $\mathbb{H}_n$ by $z \to z[{}^t u]$ gives us

$$f(z[{}^t u]) = \det(u)^k f(z) \tag{6.9}$$

Writing the Fourier series on either side gives us

$$\sum_H a(H)\exp(2\pi i\sigma(Huz\,{}^t u)) = \det(u)^k \sum_H a(H)\exp(2\pi i\sigma(Hz)) \tag{6.10}$$

We apply the cyclic property of the trace on the left hand side to get:

$$\sum_H a(H)\exp(2\pi i\sigma({}^t uHuz)) = \sum_H \det(u)^k a(H)\exp(2\pi i\sigma(Hz)) \tag{6.11}$$

Now we compare the Fourier coefficient of the $\exp(2\pi i\sigma(H[u]z))$ term on each side to get $a(H) = \det(u)^k a(H[u])$ which is equivalent to the result we wanted to prove.

□

Whilst we haven't considered the convergence of the Fourier series, we will quickly talk about why the series must run over positive semi-definite matrices H.

Lemma 55. *If the summation (6.5) were to contain an H which is not positive semi-definite, then it would diverge for $z = i\mathbb{1}$.*

We follow page 45 and 46 of Klingen's book [17] with some fleshed out explanation added via the help of Kohnen's notes [19].

Proof. For $z = i\mathbb{1}$ we have the Fourier series

$$f(i\mathbb{1}) = \sum_{H} a(H)\exp(-2\pi\sigma(H)) \tag{6.12}$$

where we assume for the sake of contradiction that the sum runs over all half-integral H with integers on the diagonal. We will assume that this series converges, and show that this implies $a(H) = 0$ for all H which are not positive semi-definite.

Since the series converges, it must be true that $|a(H)\exp(-2\pi\sigma(H))| > 1$ for only finitely many H. Otherwise there would be infinitely many terms in the series of modulus greater than 1 which would cause divergence. Thus there are finitely many H for which $|a(H)| > \exp(2\pi\sigma(H))$, and so we can find some constant $\kappa > 0$ such that $|a(H)| \leq \kappa \exp(2\pi\sigma(H))$ for all half integral H with integral diagonal.

Take an H in the summation which is not positive semi-definite. Therefore, by definition, there must exist a $\mathbf{v} \in \mathbb{R}^n$ such that $H[\mathbf{v}] < 0$. Since the function $H[\mathbf{v}]$ is continuous we can assume that $\mathbf{v}$ has rational components - and then we can scale through by a common denominator to assume that $\mathbf{v} \in \mathbb{Z}^n$ with coprime components.

As claimed by Kohnen [19], by "Gauss's Lemma", we can find a $u \in U_n^+$ such that u has $\mathbf{v}$ as its first column [1]. We can now perform a change of basis on the matirx $H \to H[u]$, which allows us to assume that $H[\mathbf{e}_1] = H_{1,1} < 0$. We can perform this change of basis without altering the Fourier coefficient $a(H)$ due to Lemma 54.

Consider the matrix

$$V_\alpha \quad := \quad \begin{pmatrix} 1 & \alpha & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in U_n^+ \ ; \ \alpha \in \mathbb{Z} \tag{6.13}$$

and let $H_\alpha := H[V_\alpha]$. Once again by Lemma 54, we have $a(H_\alpha) = a(H)$. We note that $[H_\alpha]_{2,2} = H_{1,1}\alpha^2 + H_{1,2}\alpha + H_{2,2}$ and so $\sigma(H_\alpha) = H_{1,1}\alpha^2+$ linear terms in α [19]. Therefore $\lim_{\alpha\to\infty} \sigma(H_\alpha) = -\infty$ (since $H_{1,1} < 0$).

We recall that $|a(H)| = |a(H_\alpha)| \leq \kappa \exp(2\pi\sigma(H_\alpha))$. Taking the limit of the right hand side as $\alpha \to \infty$ gives 0 and thus $a(H) = 0$ and we achieve our contradiction. The summation must be restricted to H positive semi-definite.

□

6.1.2 The Koecher Principle

We shall now discuss the Koecher Principle, which essentially shows that for $n \geq 2$ the first two conditions in definition 50 of a Siegel modular form imply the third. We will demonstrate the argument given by Kohnen [19] with a slight simplification.

Theorem 56. *Take f a holomorphic function on $\mathbb{H}_n$ with $n \geq 2$ and such that f transforms under the transformation law given in definition 50. Take also a positive number d.*

Then f is bounded on the set $\{z = x + iy \in \mathbb{H}_n : y - d\mathbb{1} \geq 0\}$.

Proof. Take f satisfying conditions (i) and (ii) in definition 50. Then by Proposition 53 we have that

$$\begin{aligned} f(z) &= \sum_H a(H)\exp(2\pi i\sigma(Hz)) = \sum_H a(H)\exp(2\pi i\sigma(H(x+iy)) \\ &= \sum_H a(H)\exp(2\pi i\sigma(Hx) - 2\pi\sigma(Hy)) \end{aligned} \tag{6.14}$$

Thus taking the absolute value and applying the triangle inequality:

$$|f(z)| \leq \sum_H |a(H)|\exp(-2\pi\sigma(Hy)) \tag{6.15}$$

The next step is missing from the work of both Klingen [17] and Kohnen [19] but it is far from trivial. We refer to the paper of Coope [9] to fill in this gap. Coope states that if A and B are positive definite matrices then $\sigma(AB) > 0$ (or ≥ 0 if A and B are positive semi-definite). We prove this in Appendix C.

Since H and $y - d\mathbb{1}$ are positive semi-definite it is clear to see that $\sigma(H(y - d\mathbb{1})) \geq 0$

By linearity of the trace we therefore have $\sigma(Hy) \geq \sigma(dH)$.

Thus $\exp(-2\pi\sigma(dH)) \geq \exp(-2\pi\sigma(Hy))$ and we can deduce:

$$|f(z)| \leq \sum_H |a(H)|\exp(-2\pi\sigma(dH)) \tag{6.16}$$

The right hand side converges, since the series $f(id\mathbb{1})$ converges absolutely as per Proposition 53 and the right hand side is just this series with the absolute value taken on all the terms, so therefore converges by the definition of absolute convergence and thus f is bounded. □

We should now convince ourselves that it follows that f is bounded on $\mathcal{F}_n$. We will require a lemma from page 30 of Klingen's book [17], although we will have to omit the proof for the sake of time.

Definition 57. *The vertical strip of positive height d is the subset*

$$V_n(d) = \{z = x + iy \in \mathbb{H}_n : \sigma(x^2) \leq d^{-1}, y - d\mathbb{1} \geq 0\} \tag{6.17}$$

Lemma 58. *There exists a positive number $d \in \mathbb{R}_{>0}$ such that $\mathcal{F}_n \subset V_n(d)$.*

Proof. See Klingen [17] page 30. □

Corollary 59. *Take f as in Theorem 56. Then f is bounded on Siegel's Fundamental Domain.*

Proof. Take d such that $\mathcal{F}_n \subset V_n(d)$ by Lemma 58. Then we have $\mathcal{F}_n \subset V_n(d) \subset \{z \in \mathbb{H}_n : y - d\mathbb{1}\}$

Since f is bounded on the set $\{z \in \mathbb{H}_n : y - d\mathbb{1}\}$ it is therefore bounded on $\mathcal{F}_n$. □

Remark 60. *In fact, the Koecher principle also works for automorphic forms transforming under the subgroup A (As per Klingen [17]). This is because throughout the discussion of Fourier series we only ever needed elements of the group A as opposed to all of* $\mathrm{Sp}(n, \mathbb{Z})$.

6.2 Siegel Modular Forms for Hecke Subgroups

6.2.1 Dirichlet Characters

In order to define the more general transformation law required for Siegel modular forms of Hecke subgroups $\Gamma_0^{(n)}(N)$ we will first need to define 'Dirichlet Characters' which are a special type of multiplicative function. Our definition is taken from the book of Ireland and Rosen titled 'A Classical Introduction to Modern Number Theory' [15].

Definition 61. *A Dirichlet Character modulo N is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that:*

- $\chi(a+N) = \chi(N)$ *for all $a \in \mathbb{Z}$. That is, χ is periodic with period N.*
- $\chi(a)\chi(b) = \chi(ab)$ *for all $a, b \in \mathbb{Z}$. That is, χ is completely multiplicative.*
- $\chi(a) = 0$ *if and only if* $\gcd(a, N) \neq 1$.

We will now introduce a group character on the Hecke Subrgoup $\Gamma_0^{(n)}(N)$, which we will call $\chi^{(n)}$. This definition is taken from Dickson [10].

Definition 62. *Given χ a Dirichlet character modulo N, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(n)}(N)$ we define a group character $\chi^{(n)}$ on $\Gamma_0^{(n)}(N)$ by the rule*

$$\chi^{(n)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(\mathrm{det(d)}) \tag{6.18}$$

It can be shown that we could also define this equivalently as $\chi^{(n)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \overline{\chi}(\mathrm{det(a)})$. [10] This is because we have $\det(\gamma) \equiv \det(a)\det(d) \equiv 1 \bmod N$. Sometimes we will drop the superscript $^{(n)}$ via abuse of notation and simply refer to the group character as χ.

6.2.2 Siegel Modular Forms of level N

The introduction of Dirichlet characters and Hecke Subgroups will allow us to define a more general type of Siegel modular form.

Definition 63. *A function $f : \mathbb{H}_n \to \mathbb{C}$ is called a Siegel modular form of degree n, weight k, level N and character χ if:*

(i) f is holomorphic

(ii) $f(\gamma\langle z\rangle) = \chi(\gamma)j(\gamma, z)^k f(z)$ for all $\gamma \in \Gamma_0^{(n)}(N)$ and $z \in \mathbb{H}_n$, where χ is a Dirichlet character modulo N.

(iii) For $n = 1$ we also require that f is holomorphic on all cusps of $\Gamma_0^{(1)}(N)$.

Remark 64. *In analogue to the case of Siegel modular forms for the full Modular group* $\mathrm{Sp}(n, \mathbb{Z})$*, it can be shown that when $n > 1$ the third condition* (iii) *follows as a consequence of the first two [10].*

Remark 65. *It can also be shown that Siegel modular forms of degree n, weight k, level N and character χ form a vector space over $\mathbb{C}$. We denote this space by $\mathcal{M}_k^{(n)}(N, \chi)$.*

Chapter 7

Siegel Theta Series

As discussed in the introduction, theta series are some of the most interesting objects in the study of classical modular forms.

Naturally this leads us to ask the question of whether theta series exist of degree $n > 1$.

In this section we shall define a more generalised version of theta series as functions on Siegel's half-space and show that these functions are indeed Siegel modular forms.

The majority of this chapter follows the paper of Andrianov and Maloletkin [5] which first laid out the proof that these functions are Siegel modular forms.

Some of their results are covered in higher generality than what is appropriate to our discussion. We will follow the proof of their first theorem in slightly lower generality which allows us to draw more obvious comparisons to classical theta series.

The paper [5] is well written, however some of the notation is cumbersome and many steps are not detailed. We shall cover the proofs in more detail in order to add clarity to their work.

7.1 Quadratic Forms

First consider an arbitrary integral positive definite quadratic form in m variables:

$$f(\mathbf{x}) = \sum_{1 \leq i \leq j \leq m} f_{ij} x_i x_j \tag{7.1}$$

where $\mathbf{x}$ is an m-component vector and $f_{ij} \in \mathbb{Z}$ for all i, j.

This quadratic form is associated to the symmetric matrix

$$F = \begin{pmatrix} 2f_{11} & \cdot & \cdot & \cdot & f_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{1m} & \cdot & \cdot & \cdot & 2f_{mm} \end{pmatrix} \tag{7.2}$$

since ${}^t\mathbf{x}F\mathbf{x} = 2f(\mathbf{x})$. Note that the leading diagonal has to have even entries since each of these entries only gets counted once, whilst all other entries are counted twice each.

Definition 66. *For a positive definite matrix F with integer entries and even diagonal (as above), the level of F is the smallest integer N such that NF^{-1} is also a positive definite matrix with integer entries and even diagonal.*

7.2 Siegel Theta Series and Representation Numbers

Definition 67. *The theta series of degree n associated with matrix F (or quadratic form f) is:*

$$\theta_F^{(n)}(z) := \sum_M \exp(\pi i\sigma({}^tMFMz)) = \sum_M \exp(2\pi i\sigma(f(M)z)) \tag{7.3}$$

where:

- *$\sigma(T)$ denotes the trace of a matrix T*
- *z is a point in Siegel's half-space $\mathbb{H}_n$*
- *The summation runs over all matrices $M \in \mathrm{M}_{m\times n}(\mathbb{Z})$ with m rows and n columns and integer entries.*

Note that for $n = 1$ we revert to the standard classical case. Recall that here we could use the Fourier coefficients of the theta series to count the number of vectors $\mathbf{x} \in \mathbb{Z}^m$ with a certain 'length' with respect to the given quadratic form.

Now for $n > 1$ we can generalise this concept. By re-writing the theta series in (7.3) with terms grouped when they have equal exponent we get:

$$\theta_F^{(n)}(z) = \sum_G r_F(G)\exp(\pi i\sigma(Gz)) \tag{7.4}$$

where G runs through all integral symmetric positive semi-definite $n \times n$ matrices with even diagonal.

This leads to the Fourier coefficients being the representation numbers:

$$r_F(G) := \#\{M \in \mathrm{M}_{m\times n}(\mathbb{Z}) : {}^tMFM = G\} \tag{7.5}$$

7.2.1 The Generalised Representation Numbers

We should ask the question as to why the G that we sum over must be positive semi-definite. Of course, if we knew that the theta series were Siegel modular forms then we could apply Proposition 53 and get this automoatically - but at this stage we don't know whether the theta series are modular forms! We should also ask whether the representation numbers $r_F(G)$ are even finite.

Lemma 68. *Take $G \in \mathrm{M}_n(\mathbb{Z})$ and take $F \in \mathrm{M}_m(\mathbb{Z})$ positive definite with even diagonal. If there exists an $M \in \mathrm{M}_{m\times n}(\mathbb{Z})$ such that ${}^tMFM = G$ then G must be positive semi-definite with even diagonal.*

The above lemma is stated as a comment by Kohnen [19] without justification. We shall detail the reasoning here.

Proof. We investigate the diagonal elements of G, given by $g_{\nu\nu}$. We have $g_{\nu\nu} = {}^tM_\nu FM_\nu$ where M_ν is the ν-th column of M. We write this in terms of the quadratic form f induced by F.

$$2f(M_\nu) = g_{\nu\nu} \tag{7.6}$$

So clearly G must have even diagonal. We now investigate the definite-ness of G.

Take $\mathbf{v} \in \mathbb{R}^n$.

$$ {}^t\mathbf{v}G\mathbf{v} = {}^t\mathbf{v}\,{}^tMFM\mathbf{v} = {}^t(M\mathbf{v})FM\mathbf{v} \tag{7.7}$$

$M\mathbf{v}$ is a vector in $\mathbb{R}^m$ which, notably, could be equal to $\mathbf{0}$ (as we make no assumption on the rank of M). Thus we can deduce by the fact that F is positive definite that ${}^t(M\mathbf{v})FM\mathbf{v} \geq 0$ and thus G is positive semi-definite. □

Lemma 69. *$r_F(G)$ is finite for all integral positive definite F with even diagonal and all integral positive semi-definite G with even diagonal.*

Kohnen [19] gives a sketch proof but does not justify any of the steps. We shall write a more complete proof.

Proof. Firstly we will prove the case when $n = 1$. In other words we assume $G \in 2\mathbb{N}$. To differentiate between dealing with full $n \times n$ matrices and integers we will here let $g := G$.

Since F is positive definite, it is diagonalisable with positive eigenvalues [26]. Now consider the general case where $g \in \mathbb{Z}$ and we want to represent it as ${}^t\mathbf{x}F\mathbf{x} = g$ with $\mathbf{x} \in \mathbb{R}^m$. Without going into the analytic details, it should be simple to convince oneself that the set $M_g := \{\mathbf{x} \in \mathbb{R}^m : {}^t\mathbf{x}F\mathbf{x} = g\}$ is compact (since it is the boundary of a ball with respect to the metric induced by F).

We note that $M_g(\mathbb{Z}) := \{\mathbf{x} \in \mathbb{Z}^m : {}^t\mathbf{x}F\mathbf{x} = g\} = M_g \cap \mathbb{Z}^m$ which is an intersection of a compact set with a discrete set. The following argument was given by a user on Stack Exchange [2]. Suppose that this set is infinite in size. Then by the compactness of M_g, we know that every infinite subset of M_g contains an accumulation point. Thus we deduce that $M_g(\mathbb{Z})$ would contain an accumulation point, say $\mathbf{x}'$. Then every neighbourhood of $\mathbf{x}'$ would contain other points in $M_g(\mathbb{Z})$, and thus $\mathbb{Z}^m$, but this contradicts the fact that $\mathbb{Z}^m$ is discrete. Therefore we conclude that the set $M_g(\mathbb{Z})$ is finite and so $r_F(g)$ is a finite number.

We now move to the case of a general n. Take a representation of G by F: ${}^tMFM = G$ for some $M \in \mathrm{M}_{m\times n}(\mathbb{Z})$. We note that the diagonal elements of of G can be written as $g_{\nu\nu} = {}^tM_\nu FM_\nu$ where M_ν is the ν-th column of M. Since $M_\nu \in \mathbb{Z}^m$, this is exactly a way of representing an integer $g_{\nu\nu}$ by the matrix F, so by the above we know there are a finite number of choices for M_ν.

Since there are a finite number of ways to choose M_ν this means there are a finite number of ways to choose the matrices M. Hence the representation number $r_F(G)$ is finite. □

7.3 Theta Series are Siegel Modular Forms

From the classical case $n = 1$, most of the results which allow us to determine formulae for the representation numbers are given as a result of knowing that the theta series are modular forms of weight $m/2$ for some Hecke subgroup.

Our goal will be to prove that in general the theta series of degree n are Siegel modular forms. This may allow us to apply results from the theory of Siegel modular forms to give explicit formulae or asymptotics for the representation numbers $r_F(G)$.

We will follow the proof of theorem 1 given by Andrianov and Maloletkin. We will prove a statement in lower generality than what is given in their paper [5] since we won't need the whole theorem for our discussion and it will save us some time.

Andrianov and Maloletkin prove a more general transformation law for the 'generalised theta series'

$$\theta_F^{(n)}(z; X, Y) = \sum_{M\in \mathrm{M}_{m\times n}(\mathbb{Z})} \exp(\pi i\sigma(zF[M-Y]) + 2\pi i\sigma({}^tMX) - \pi i\sigma({}^tXY)) \tag{7.8}$$

Here, X and Y are $m \times n$ complex matrices. We prove the special case where $X = Y = 0$ which is exactly the theta series given in (7.3). The general case is no more complicated to prove and uses all of the same techniques as the proof of the special case.

7.3.1 The Theta Function

In order to prove that theta series are Siegel modular forms, we will first require a similar result about a slightly different object called the 'theta function'. This is stated and proved by Eichler in his book titled 'Introduction to the theory of algebraic numbers and functions' [11].

Definition 70. *The theta function of degree n is*

$$\vartheta_n(\tilde{z}) = \sum_{\tilde{\mathbf{v}} \in \mathbb{Z}^n} \exp(\pi i \tilde{z}[\tilde{\mathbf{v}}]) \tag{7.9}$$

where the argument $\tilde{z}$ is in Siegel's half-space of degree n, $\mathbb{H}_n$.

The theta function is essentially a generalisation of the well known Jacobi theta series [12] to the n-dimensional Siegel's half-space.

Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z})$. Via Symplecticity (2.19) we have that ${}^t ca$ and ${}^t bd$ are symmetric matrices. Moreover they have integral entries.

Lemma 71. *The set $\Theta^{(n)} := \{\gamma \in \mathrm{Sp}(n, \mathbb{Z}) : {}^t ca \text{ and } {}^t bd \text{ have even diagonal}\}$ is a subgroup of Γ_n called the 'theta group'*

For the sake of time we will not prove the above lemma but it should be fairly simple to see how one would deduce this by expanding out the matrix multiplication.

Theorem 72. *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Theta^{(n)}$ and $\tilde{z} \in \mathbb{H}_n$ we have:*

$$\vartheta_n(\gamma\langle\tilde{z}\rangle) = \chi(\gamma) \det(cz + d)^{\frac{1}{2}} \vartheta_n(\tilde{z}) \tag{7.10}$$

where the character χ is a function on $\Theta^{(n)}$ with values in the group of eighth roots of unity.

The above transformation law is proven by Eichler [11]. We will not have time to cover the proof and it will be more informative for us to see how this theorem can be applied to Siegel theta series in general. Notice that for $n = 1$ we have a Jacobi theta series and recall we can prove the transformation formula by using the Poisson summation formula. [12]

Here we should clarify the meaning of the 'square root' in the above, since in general $\det(cz+d)$ is a complex number. As per the explanation on page 41 of Eichler, we take the positive square root when $\det(cz + d)$ is real, and in general the sign of the square root is found by taking the analytic continuation of $\det(cz + d)^{\frac{1}{2}}$ to $\mathbb{H}_n$.

7.3.2 Theta Series

Now that we have the transformation law for the theta function (7.10), we will be able to follow the work of Andrianov and Maloletkin [5] to prove the following theorem.

Theorem 73. *The theta series $\theta_F^{(n)}$ defined in (7.3) satisfies the following transformation law (or functional equation):*

$$\theta_F^{(n)}(\gamma\langle z\rangle) = \chi_F^{(n)}(\gamma)\det(cz+d)^{\frac{m}{2}}\theta_F^{(n)}(z) \tag{7.11}$$

for all $\gamma \in \Gamma_0^{(n)}(N)$, $z \in \mathbb{H}_n$, where N is the level of the matrix F and $\chi_F^{(n)}(\gamma)$ is some eighth root of unity.

Proof. Our proof strategy will be to write the theta series $\theta_F^{(n)}(z)$ as the value of the theta function $\vartheta_l(\tilde{z})$ for a suitable choice of degree l and argument $\tilde{z}$. We will then embed the group $\Gamma_0^{(n)}(N)$ into the theta group $\Theta^{(l)}$ and this will allow us to apply the theta function transformation law (7.10) proven by Eichler.

This proof will also make use of a number of results about tensor products of matrices. These will be familiar from courses in linear algebra, representation theory or quantum computing. A list of tensor product properties can be found in Appendix B.

Firstly, we will pick our choice of degree for ϑ_l to be $l = mn$ where m is the 'dimension' of the matrix F and n is the degree of the Siegel's half-space for the theta series $\theta_F^{(n)}(z)$.

Next we will make a choice for the $\tilde{z}$. Since the degree of the theta function is mn, this means that we have $\tilde{z} \in \mathbb{H}_{mn}$.

We define a mapping from $\mathbb{H}_n$ to $\mathbb{H}_{mn}$ by

$$\begin{aligned} \phi : \mathbb{H}_n &\rightarrow \mathbb{H}_{mn} \\ z &\mapsto F \otimes z =: \tilde{z} \end{aligned} \tag{7.12}$$

So we 'encode' the matrix F into the argument of the theta function $\tilde{z}$ via this tensor product.

We ought to check that $F \otimes z \in \mathbb{H}_{mn}$. Firstly, z is an $n \times n$ matrix and F is an $m \times m$ matrix, so by the definition of the tensor product we must have that $F \otimes z$ is an $mn \times mn$ matrix. To show symmetry we apply the property of tensor products that ${}^t(F \otimes z) = {}^tF \otimes {}^tz$. Since F and z are both symmetric we deduce that ${}^t(F \otimes z) = F \otimes z$. To show that the imaginary part is positive definite we first note that F is an entirely real matrix, so $\mathrm{Im}(F \otimes z) = F \otimes \mathrm{Im}(z) = F \otimes y$. We now note that F and y are both positive definite, so we can apply the tensor product property that $F > 0, y > 0 \implies F \otimes y > 0$. So indeed $\tilde{z} = F \otimes z \in \mathbb{H}_{mn}$.

Our next step is to consider the summations in the theta series and theta function respectively. In the theta series (7.3) our summation runs over $M \in \mathrm{M}_{m \times n}(\mathbb{Z})$ whereas in the theta function the summation runs over $\tilde{\mathbf{v}} \in \mathbb{Z}^{mn}$. As we did for the arguments z and $\tilde{z}$, we will construct a mapping between the two settings. In fact we will construct a more general mapping between $\mathrm{M}_{m \times n}(\mathbb{C})$ and $\mathbb{C}^{mn}$. Firstly, for $M \in \mathrm{M}_{m \times n}(\mathbb{C})$ we denote by M_i the i-th column of M for $1 \leq i \leq n$.

$$\begin{aligned} \varphi : \mathrm{M}_{m \times n}(\mathbb{C}) &\rightarrow \mathbb{C}^{mn} \\ M = (M_1, ..., M_n) &\mapsto \begin{pmatrix} M_1 \\ \cdot \\ \cdot \\ \cdot \\ M_n \end{pmatrix} =: \tilde{\mathbf{v}} \end{aligned} \tag{7.13}$$

so essentially what we're doing is taking the columns of M and 'stacking' them on top of each other into one long column vector $\tilde{\mathbf{v}}$. It should be fairly clear to see that this mapping is one-to-one.

We will now prove an important identity involving these mappings. Take $\tilde{z} = \phi(z)$ and $\tilde{\mathbf{v}} = \varphi(M)$, for $z \in \mathbb{H}_n$ and $M \in \mathrm{M}_{m\times n}(\mathbb{C})$.

$$\begin{aligned}
\tilde{z}[\tilde{\mathbf{v}}] &= ({}^tM_1, ..., {}^tM_n) \begin{pmatrix} Fz_{11} & \cdot & \cdot & \cdot & Fz_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Fz_{n1} & \cdot & \cdot & \cdot & Fz_{nn} \end{pmatrix} \begin{pmatrix} M_1 \\ \cdot \\ \cdot \\ \cdot \\ M_n \end{pmatrix} \\
&= \sum_{1\leq i,j\leq n} z_{ij}\, {}^tM_iFM_j = \sum_{1\leq i,j\leq n} z_{ij}[F[M]]_{ij} \\
&= \sum_{1\leq j\leq n} [zF[M]]_{jj} = \sigma(zF[M]) = \sigma(F[M]z) \qquad (7.14)
\end{aligned}$$

The final equality follows from the cyclic property of the trace. In the above we make the deductions in a slightly clearer way to what is presented in Andrianov and Maloletkin's paper [5]. The step $\sum z_{ij}\, {}^tM_iFM_j = \sum z_{ij}[F[M]]_{ij}$ follows from noting that ${}^tM_iFM_j$ is the ij-th entry of the $n \times n$ matrix $F[M]$. For the final two steps we notice that the summation is a matrix multiplication and trace calculation all in one.

We can now apply this identity to the definition of the theta series:

$$\begin{aligned}
\theta_F^{(n)}(z) &= \sum_{M\in \mathrm{M_{m\times n}}(\mathbb{Z})} \exp(\pi i\sigma(F[M]z)) = \sum_{M\in \mathrm{M_{m\times n}}(\mathbb{Z})} \exp(\pi i(\phi(z)[\varphi(M)])) \\
&= \sum_{\tilde{\mathbf{v}}\in\mathbb{Z}^{mn}} \exp(\pi i\tilde{z}[\tilde{\mathbf{v}}]) = \vartheta_{mn}(\tilde{z}) = \vartheta_{mn}(\phi(z)) \qquad (7.15)
\end{aligned}$$

Thus we have completed our first goal: to show that $\theta_F^{(n)}(z) = \vartheta_{mn}(\tilde{z})$. However we must still show the transformation law (7.11). For this we will now embed the modular group for the theta series into the theta group. Namely the group $\Gamma_0^{(n)}(N)$ into the group $\Theta^{(mn)}$.

We once again define a mapping (a homomorphism in fact) from one 'world' to the other.

$$\begin{aligned}
\Phi : \Gamma_0^{(n)}(N) &\rightarrow \Theta^{(mn)} \\
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} \mathbb{1}_m \otimes a & F\otimes b \\ F^{-1}\otimes c & \mathbb{1}_m \otimes d \end{pmatrix} =: \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \tilde{\gamma} \qquad (7.16)
\end{aligned}$$

Note that $\tilde{\gamma}$ is an $mn \times mn$ matrix.

Here we see that we are once again encoding our matrix F in some way. We will now want to show that $\tilde{\gamma} \in \Theta^{(mn)} \leqslant \mathrm{Sp}(mn, \mathbb{Z})$.

We first verify that $\tilde{\gamma}$ has integer entries. Then we will check that it is in the Symplectic group, before finally showing that it is in the theta group.

Since a, b, c, d and F are all integral matrices, it is clear that $\tilde{a}, \tilde{b}, \tilde{d}$ are all integral matrices too. We need to check $\tilde{c} = F^{-1} \otimes c$ though.

We recall that $\gamma \in \Gamma_0^{(n)}(N) \implies c \equiv 0 \bmod N$. So there exists some integral matrix $\hat{c}$ such that $c = N\hat{c}$. We can therefore write $\tilde{c} = F^{-1} \otimes N\hat{c} = NF^{-1} \otimes \hat{c}$.

We know from the statement of the theorem that N was the level of the matrix F, which we recall means that NF^{-1} has integer entries. Thus $\tilde{c}$ must have integer entries.

We now want to show that $\tilde{\gamma} \in \mathrm{Sp}(mn, \mathbb{Z})$. We will take a slightly different approach to the paper of Andrianov and Maloletkin [5] where they check the symplecticity conditions directly. We will go back to the definition of the symplectic group and check that $\tilde{\gamma}$ satisfies the defining condition that $j[\tilde{\gamma}] = j$, where here $j = \begin{pmatrix} 0 & \mathbb{1}_{mn} \\ -\mathbb{1}_{mn} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_m \otimes \mathbb{1}_n \\ -\mathbb{1}_m \otimes \mathbb{1}_n & 0 \end{pmatrix}$.

$$\begin{aligned}
j[\tilde{\gamma}] = {}^t\tilde{\gamma} j \tilde{\gamma} &= \begin{pmatrix} {}^t\tilde{a} & {}^t\tilde{c} \\ {}^t\tilde{b} & {}^t\tilde{d} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_m \otimes \mathbb{1}_n \\ -\mathbb{1}_m \otimes \mathbb{1}_n & 0 \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \\
&= \begin{pmatrix} {}^t(\mathbb{1}_m \otimes a) & {}^t(F^{-1} \otimes c) \\ {}^t(F \otimes b) & {}^t(\mathbb{1}_m \otimes d) \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_m \otimes \mathbb{1}_n \\ -\mathbb{1}_m \otimes \mathbb{1}_n & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_m \otimes a & F \otimes b \\ F^{-1} \otimes c & \mathbb{1}_m \otimes d \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{1}_m \otimes {}^ta & F^{-1} \otimes {}^tc \\ F \otimes {}^tb & \mathbb{1}_m \otimes {}^td \end{pmatrix} \begin{pmatrix} F^{-1} \otimes c & \mathbb{1}_m \otimes d \\ -\mathbb{1}_m \otimes a & -F \otimes b \end{pmatrix} \\
&= \begin{pmatrix} F^{-1} \otimes {}^tac - F^{-1} \otimes {}^tca & \mathbb{1}_m \otimes {}^tad - F^{-1}F \otimes {}^tcb \\ FF^{-1} \otimes {}^tbc - \mathbb{1}_m \otimes {}^tda & F \otimes {}^tbd - F \otimes {}^tdb \end{pmatrix} \\
&= \begin{pmatrix} F^{-1} \otimes ({}^tac - {}^tca) & \mathbb{1}_m \otimes ({}^tad - {}^tcb) \\ \mathbb{1}_m \otimes ({}^tbc - {}^tda) & F \otimes ({}^tbd - {}^tdb) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathbb{1}_m \otimes \mathbb{1}_n \\ -\mathbb{1}_m \otimes \mathbb{1}_n & 0 \end{pmatrix} = j
\end{aligned} \tag{7.17}$$

In the last step we have applied the symplecticity conditions for γ. So indeed $\tilde{\gamma} \in \mathrm{Sp}(mn, \mathbb{Z})$. Finally we must prove that $\tilde{\gamma} \in \Theta^{(mn)}$ - so we must show that ${}^t\tilde{c}\tilde{a}$ and ${}^t\tilde{b}\tilde{d}$ have even entries on the diagonal.

We see that ${}^t\tilde{b}\tilde{d} = F \otimes {}^tbd$. If we let ν_{ij} be the ij-th entry of the matrix ${}^tbd$ we see that

$$ {}^t\tilde{b}\tilde{d} = F \otimes {}^tbd = \begin{pmatrix} F\nu_{11} & \cdot & \cdot & \cdot & F\nu_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F\nu_{n1} & \cdot & \cdot & \cdot & F\nu_{nn} \end{pmatrix} \tag{7.18}$$

Clearly the ν_{ii} are integers for all i, and we know that F has even entries on the diagonal. Therefore we see that $F\nu_{ii}$ has an even diagonal for all i and thus the full matrix ${}^t\tilde{b}\tilde{d}$ has even diagonal.

Now we consider the matrix ${}^t\tilde{c}\tilde{a} = F^{-1} \otimes {}^tca$. We can write this as ${}^t\tilde{c}\tilde{a} = NF^{-1} \otimes {}^t(N^{-1}c)a = NF^{-1} \otimes {}^t\hat{c}a$. N was the level of F - so NF^{-1} has an even diagonal and $\hat{c}$ and a are both integral matrices. So we can apply the same logic as we did for ${}^t\tilde{b}\tilde{d}$ to say that ${}^t\tilde{c}\tilde{a}$ has an even diagonal. So indeed $\Phi(\gamma) = \tilde{\gamma} \in \Theta^{(mn)}$ and we have embedded a copy of $\Gamma_0^{(n)}(N)$ into $\Theta^{(mn)}$.

Now we can apply Theorem 72 - the transformation law for the theta function.

$$\vartheta_{mn}(\tilde{\gamma}\langle\tilde{z}\rangle) = \chi(\tilde{\gamma})\det(\tilde{c}\tilde{z} + \tilde{d})^{\frac{1}{2}}\vartheta_{mn}(\tilde{z}) \tag{7.19}$$

where $\tilde{\gamma} = \Phi(\gamma)$ for an arbitrary $\gamma \in \Gamma_0^{(n)}(N)$ and $\tilde{z} = \phi(z)$ for an arbitrary $z \in \mathbb{H}_n$.

We use the properties of tensor products and definitions of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ and $\tilde{z}$ to perform some manipulations:

$$\begin{aligned}\tilde{a}\tilde{z}+\tilde{b} &= (\mathbb{1}_m \otimes a)(F\otimes z)+F\otimes b = F\otimes(az+b)\\ \tilde{c}\tilde{z}+\tilde{d} &= (F^{-1}\otimes \tilde{c})(F\otimes z)+\mathbb{1}_m\otimes d = \mathbb{1}_m\otimes(cz+d)\end{aligned} \tag{7.20}$$

Thus we can deduce

$$\begin{aligned}\tilde{\gamma}\langle\tilde{z}\rangle &= (\tilde{a}\tilde{z}+\tilde{b})(\tilde{c}\tilde{z}+\tilde{d})^{-1} = (F\otimes(az+b))(\mathbb{1}_m\otimes(cz+d))^{-1}\\ &= (F\otimes(az+b))(\mathbb{1}_m\otimes(cz+d)^{-1}) = F\otimes(az+b)(cz+d)^{-1}\\ &= F\otimes\gamma\langle z\rangle = \phi(\gamma\langle z\rangle)\end{aligned} \tag{7.21}$$

and via the property that $\det(A_{m\times m}\otimes B_{n\times n}) = \det(A)^n\cdot\det(B)^m$ we also get

$$\det(\tilde{c}\tilde{z}+\tilde{d}) = \det(\mathbb{1}_m\otimes(cz+d)) = \det(\mathbb{1}_m)^n\cdot\det(cz+d)^m = \det(cz+d)^m \tag{7.22}$$

We can then plug all the above identities into (7.19) and get:

$$\vartheta_{mn}(\phi(\gamma\langle z\rangle)) = \chi(\tilde{\gamma})\det(cz+d)^{\frac{m}{2}}\vartheta_{mn}(\phi(z)) \tag{7.23}$$

We now recall that we had previously shown $\theta_F^{(n)}(z) = \vartheta_{mn}(\phi(z))$, and we define the character of $\Gamma_0^{(n)}(N)$ to be the one induced by $\Theta^{(mn)}$:

$$\chi_F^{(n)}(\gamma) := \chi(\tilde{\gamma}) = \chi(\Phi(\gamma)) \tag{7.24}$$

We can then plug the above into (7.23) to get the desired transformation law

$$\theta_F^{(n)}(\gamma\langle z\rangle) = \chi_F^{(n)}(\gamma)\det(cz+d)^{\frac{m}{2}}\theta_F^{(n)}(z) \tag{7.25}$$

□

7.3.3 Convergence

So we have the correct transformation law for theta series to be a Siegel modular form, but we also need to check that the theta series are actually holomorphic. It is enough to show that the series (7.3) converges on certain subsets of $\mathbb{H}_n$. We will follow the proof on page 6 of the book of Andrianov and Zhuralev [6], with the gaps filled in.

Proposition 74. $\theta_F^{(n)}(z)$ *converges absolutely and uniformly on any subset of* $\mathbb{H}_n$ *of the form*

$$\mathbb{H}_n(\varepsilon) := \{z = x+iy\in\mathbb{H}_n : y-\varepsilon\mathbb{1}\geq 0\}\ ;\ \varepsilon>0 \tag{7.26}$$

In other words, for all z *such that* $y-\varepsilon\mathbb{1}$ *is positive semi-definite.*

It is clear to see that the above proposition implies locally uniform convergence on the whole domain $\mathbb{H}_n$, and thus holomorphicity.

Proof. We take absolute value of the theta series and apply the triangle inequality to see that

$$|\theta_F^{(n)}(z)| \leq \sum_M |\exp(\pi i\sigma({}^tMFMx))\exp(\pi i\sigma(i\,{}^tMFMy))| = \sum_M \exp(-\pi\sigma({}^tMFMy)) \tag{7.27}$$

We let λ be the smallest eigenvalue of the matrix F. Then the matrix $F - \lambda \mathbb{1}$ must be positive semi-definite.

Therefore for any $M \in \mathrm{M}_{m\times n}(\mathbb{R})$ we must have ${}^tM(F - \lambda\mathbb{1})M = {}^tMFM - \lambda\, {}^tMM$ positive semi-definite. Thus $\sigma({}^tMFM) \geq \sigma(\lambda\, {}^tMM)$.

We also have that $y - \varepsilon\mathbb{1}$ is positive semi-definite so we must have that $\sigma({}^tMFM(y - \varepsilon\mathbb{1})) \geq 0$ by the result of Coope [9] (See Appendix C). Hence $\sigma({}^tMFMy) \geq \sigma(\varepsilon\, {}^tMFM)$.

Putting these two trace inequalities together gives us

$$\sigma({}^tMFMy) \geq \sigma(\varepsilon\, {}^tMFM) \geq \varepsilon\lambda\sigma({}^tMM) \tag{7.28}$$

We can apply this to the theta series

$$|\theta_F^{(n)}(z)| \leq \sum_M \exp(-\pi\sigma({}^tMFMy)) \leq \sum_M \exp(-\pi\varepsilon\lambda\sigma({}^tMM)) \tag{7.29}$$

The above 'majorization' is stated by Andrianov and Zhuralev [6] without the accompanying explanation. The book also leaves out every step of the following calculation.

Note that $[{}^tMM]_{ij} = \sum_{k=1}^n M_{ki}M_{kj}$ and thus

$$\sigma({}^tMM) = \sum_{i=1}^n [{}^tMM]_{ii} = \sum_{i=1}^n \sum_{k=1}^n M_{ki}^2 = \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}} \tag{7.30}$$

where we define $\tilde{\mathbf{v}} = \varphi(M) \in \mathbb{Z}^{mn}$ as in the proof of Theorem 73. So we have that

$$|\theta_F^{(n)}(z)| \leq \sum_{\tilde{\mathbf{v}}\in\mathbb{Z}^{mn}} \exp(-\pi\varepsilon\lambda\tilde{\mathbf{v}}\cdot\tilde{\mathbf{v}}) = \sum_{v_1\in\mathbb{Z}} \sum_{v_2\in\mathbb{Z}} \dots \sum_{v_{mn}\in\mathbb{Z}} \exp(-\pi\varepsilon\lambda(v_1^2 + v_2^2 + \dots + v_{mn}^2)) \tag{7.31}$$

Via a combinatorial argument (Recall the classical Jacobi theta series raised to some power [12]) we see that the right hand side reduces to:

$$|\theta_F^{(n)}(z)| \leq \left(\sum_{t\in\mathbb{Z}} \exp(-\pi\varepsilon\lambda t^2)\right)^{mn} \tag{7.32}$$

The series inside the brackets is convergent by the following argument from classical modular forms seen in the notes of Funke [12] (Proof of lemma 6.3)

$$\sum_{t\in\mathbb{Z}} \exp(-\pi\varepsilon\lambda t^2) = 1 + 2\sum_{t=1}^{\infty} \exp(-\pi\varepsilon\lambda t^2) \leq 1 + 2\sum_{t=1}^{\infty} \exp(-\pi\varepsilon\lambda t) = 1 + 2\frac{e^{-\pi\varepsilon\lambda}}{1 - e^{-\pi\varepsilon\lambda}} \tag{7.33}$$

□

Corollary 75. *The theta series of degree n associated with an $m \times m$ matrix F of level N is a Siegel modular form of degree n, level N, and weight $m/2$:*

$$\theta_F^{(n)} \in \mathcal{M}_{m/2}^{(n)}(N, \chi) \tag{7.34}$$

for some character χ.

7.4 Determining The Character

Our next challenge will be to determine the exact character $\chi_F^{(n)}(\gamma)$. Andrianov and Maloletkin find this function when m is even and we shall follow their proof whilst filling in numerous gaps in their working. We will follow theorem 3 from their paper [5].

The case of level $N = 1$ was proven separately by Witt [28], and the degree $n = 1$ case is also known and a proof can be seen in Eichler's book [11] (Page 51). Andrianov and Maloletkin [5] therefore focus on the cases of higher degree and level. We shall not dedicate any time to the proofs of the 'base' cases of $N = 1$ or $n = 1$, since it will be more enlightening for our discussion of Siegel theta series in their full generality to focus on how to calculate the character for higher level and degree.

7.4.1 The Inversion Formula

In order to prove the character theorem we will need to make use of an 'inversion formula' for the fully generalised theta series (7.8). This is stated as a lemma in the paper of Andianov and Maloletkin but not proven.

Proposition 76. *Let F be an $m \times m$ real, symmetric, positive definite matrix, $z \in \mathbb{H}_n$, $X, Y \in$ $\mathrm{M}_{m\times n}(\mathbb{C})$. Then*

$$\theta_{F^{-1}}^{(n)}(-z^{-1}; Y, -X) = \det(F)^{\frac{n}{2}} \det(-iz)^{\frac{m}{2}} \theta_F^{(n)}(z; X, Y) \tag{7.35}$$

Note that in the above, we no longer require F to have integer entries and instead we only need it to be real. This is because there is no guarantee that F^{-1} has integer entries in general. Of course this change has no effect on the convergence of the theta series.

We shall give a sketch proof of the above using analogous ideas to the proof of Theorem 73. Andrianov and Maloletkin [5] hint at an idea for a proof but do not give it.

Proof. We first introduce the 'generalised theta function' [5][11]. Let $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}} \in \mathbb{C}^n$ be column vectors.

$$\vartheta_n(\tilde{z}; \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) := \sum_{\tilde{\mathbf{v}} \in \mathbb{Z}^n} \exp(\pi i \tilde{z}[\tilde{\mathbf{v}} - \tilde{\mathbf{Y}}] - 2\pi i\, {}^t\tilde{\mathbf{v}}\tilde{\mathbf{X}} - \pi i\, {}^t\tilde{\mathbf{X}}\tilde{\mathbf{Y}}) \tag{7.36}$$

This function comes equipped with its own inversion formula as proven in the book of Eichler [11].

$$\vartheta_n(-\tilde{z}^{-1}; \tilde{\mathbf{Y}}, -\tilde{\mathbf{X}}) = \det(-i\tilde{z})^{\frac{1}{2}} \vartheta_n(\tilde{z}; \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \tag{7.37}$$

Using the same technique as in the proof of Theorem 73, we can write the generalised theta series as the value of the generalised theta function for a choice of arguments. The details can be found in the paper of Andrianov and Maloletkin [5].

$$\theta_F^{(n)}(z, X, Y) = \vartheta_{mn}(\tilde{z}; \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) \tag{7.38}$$

where $\tilde{z} = \phi(z) = F \otimes z$, $\tilde{\mathbf{X}} = \varphi(X)$ and $\tilde{\mathbf{Y}} = \varphi(Y)$ where ϕ and φ are the mappings defined in the proof of Theorem 73. We note that $\tilde{z}^{-1} = F^{-1} \otimes z^{-1}$ by the properties of tensor products.

We can substitute all of the above into the generalised theta function inversion formula to get

$$\theta_{F^{-1}}^{(n)}(-z^{-1}; Y, -X) = \det(-iF \otimes z)^{\frac{1}{2}} \theta_F^{(n)}(z; X, Y) \tag{7.39}$$

We then apply the properties of the determinant of a tensor product to split up the determinant and get the required result.

$$\theta^{(n)}_{F^{-1}}(-z^{-1}; Y, -X) = \det(F)^{\frac{n}{2}} \det(-iz)^{\frac{m}{2}} \theta^{(n)}_F(z; X, Y) \tag{7.40}$$

□

7.4.2 Character Theorem Part 1: Reducing to a Gauss Sum

We will now prove the third theorem from the paper of Andrianov and Maloletkin [5] - following their argument whilst providing extra details to better explain some of their steps.

The proof will be done in two parts. The first will be to deduce that that character is equal to a 'Gauss Sum', and then the second will be to evaluate the sum and show that the character is a generalised Legendre Symbol.

In the following we will assume m is even and that $N > 1$. The case when $N = 1$ is slightly different - indeed there is a step in the first proposition where we make use of the fact that $N \neq 1$. This special case is proven in the paper of Witt [28] which is written in German:

Theorem 77. *Take $\chi^{(n)}_F$ to be the function of the group $\Gamma^{(n)}_0(N)$ as in (7.11). When $N = 1$ we have $\chi^{(n)}_F(\gamma) = 1$ for all $\gamma \in \Gamma^{(n)}_0(1) = \mathrm{Sp}(n, \mathbb{Z})$.*

So from here on we will assume $N > 1$. We must also assume in what follows that m (the dimension of F) is even.

We begin our task of determining the character by proving the following proposition - stated as lemma 4 in the paper of Andrianov and Maloletkin [5].

Proposition 78. *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(n)}_0(N)$. Let $l \in \mathbb{Z}$ be such that ld^{-1} is an integer-valued matrix and let m be even. Then:*

$$\chi^{(n)}_F(\gamma) = |\det(d)^{\frac{m}{2}}|\ l^{-mn} \sum_R \exp(\pi i \sigma(bd^{-1}F[R])) \tag{7.41}$$

where here the summation runs over all matrices $R \in \mathrm{M}_{m\times n}(\mathbb{Z})$ where the components of R are reduced modulo l. In other words: $0 \leq R_{ij} \leq l-1$. We will refer to this in shorthand by '$R \bmod l$'.

The proof follows Andrianov and Maloletkin [5] but there are plenty of points where we must provide additional details to fully justify the steps taken.

Proof. In the functional equation (7.11) we set $z = iT$ where T is some real positive definite $n \times n$ matrix.

$$\begin{aligned} \theta^{(n)}_F(\gamma\langle z\rangle) &= \det(cz+d)^{\frac{m}{2}} \chi^{(n)}_F(\gamma)\theta^{(n)}_F(z) \\ \implies \theta^{(n)}_F(\gamma\langle iT\rangle) &= \det(ciT+d)^{\frac{m}{2}} \chi^{(n)}_F(\gamma)\theta^{(n)}_F(iT) \\ \implies \det(ciT+d)^{-\frac{m}{2}}\theta^{(n)}_F(\gamma\langle iT\rangle) &= \chi^{(n)}_F(\gamma)\theta^{(n)}_F(iT) \\ \implies \det(T)^{\frac{m}{2}}\det(ciT+d)^{-\frac{m}{2}}\theta^{(n)}_F(\gamma\langle iT\rangle) &= \det(T)^{\frac{m}{2}}\chi^{(n)}_F(\gamma)\theta^{(n)}_F(iT) \end{aligned} \tag{7.42}$$

where in the final line we multiply both sides by $\det(T)^{\frac{m}{2}}$. We will now take the limit on both sides as $T \to 0$.

$$\begin{aligned}
\lim_{T\to 0} \det(T)^{\frac{m}{2}} \det(ciT+d)^{-\frac{m}{2}} \theta_F^{(n)}(\gamma\langle iT\rangle) &= \lim_{T\to 0} \det(T)^{\frac{m}{2}} \chi_F^{(n)}(\gamma)\theta_F^{(n)}(iT) \\
\lim_{T\to 0} \det(T)^{\frac{m}{2}} \det(ciT+d)^{-\frac{m}{2}} \theta_F^{(n)}(\gamma\langle iT\rangle) &= \chi_F^{(n)}(\gamma) \lim_{T\to 0} \det(T)^{\frac{m}{2}} \theta_F^{(n)}(iT) \qquad (7.43)
\end{aligned}$$

This equation will be very important and we will refer back to it a lot. We will now proceed to compute the limit on each side.

We will first calculate the right hand side. We begin by applying the inversion formula with $z = iT$, $X = Y = 0$, noting that $-iz = T$ and $-z^{-1} = iT^{-1}$.

$$\theta_F^{(n)}(iT) = \det(F)^{-\frac{n}{2}} \det(T)^{-\frac{m}{2}} \theta_{F^{-1}}^{(n)}(iT^{-1}) \qquad (7.44)$$

We now plug this into the right hand side of (7.43)

$$\begin{aligned}
\text{RHS} &= \chi_F^{(n)}(\gamma) \lim_{T\to 0} \det(T)^{\frac{m}{2}} \det(F)^{-\frac{n}{2}} \det(T)^{-\frac{m}{2}} \theta_{F^{-1}}^{(n)}(iT^{-1}) \\
&= \chi_F^{(n)}(\gamma) \det(F)^{-\frac{n}{2}} \lim_{T\to 0} \theta_{F^{-1}}^{(n)}(iT^{-1}) \qquad (7.45)
\end{aligned}$$

We now need to find the limit $\lim_{T\to 0} \theta_{F^{-1}}^{(n)}(iT^{-1})$, which is not shown in the paper [5].

$$\begin{aligned}
\lim_{T\to 0} \theta_{F^{-1}}^{(n)}(iT^{-1}) &= \lim_{T\to 0} \sum_M \exp(\pi i \sigma({}^tMF^{-1}MiT^{-1})) \\
&= \lim_{T\to 0} \sum_M \exp(-\pi \sigma({}^tMF^{-1}MT^{-1})) \\
&= 1 + \lim_{T\to 0} \sum_{M\neq 0} \exp(-\pi \sigma({}^tMF^{-1}MT^{-1})) \qquad (7.46)
\end{aligned}$$

Where in the last line we separate out the $M = 0$ term which forces the exponent to be 0. Since F^{-1} is positive definite, so is ${}^tMF^{-1}M$. We also have that T^{-1} is positive definite - so by the result in the paper of Coope [9] (See Appendix C) it is true that when $M \neq 0$ that $\sigma({}^tMF^{-1}MT^{-1}) > 0$. Thus the exponent is negative and so all terms go to 0 as $T \to 0$.

Thus we can see that

$$\text{RHS} = \chi_F^{(n)}(\gamma)\det(F)^{-\frac{n}{2}} \qquad (7.47)$$

We now move onto computing the left hand side of (7.43), which is more difficult.

We first recall via Lemma 27 that $N > 1 \implies \det(d) \neq 0$ and thus d is invertible.

We now let $T_1 := {}^td^{-1}T(icT+d)^{-1}$ and claim that $\gamma\langle iT\rangle = bd^{-1} + iT_1$. This fact is stated in the paper [5] but requires some work.

First we note that from symplecticity we have

$$\begin{aligned}
{}^tad - {}^tcb &= \mathbb{1} \\
\implies {}^tda - {}^tbc &= \mathbb{1} \\
\implies {}^tda &= \mathbb{1} + {}^tbc \\
\implies a &= {}^td^{-1} + {}^td^{-1}\,{}^tbc \\
\implies a &= {}^td^{-1} + {}^t(bd^{-1})c \qquad (7.48)
\end{aligned}$$

Also via symplecticity we have that ${}^t bd = {}^t db$. Thus ${}^t(bd^{-1}) = bd^{-1}$. So plugging this into the above equation gives us $a = bd^{-1}c + {}^t d^{-1}$. Using this fact, we have

$$\begin{aligned}(aiT+b) &= (bd^{-1}c + {}^t d^{-1})iT + b \\ &= ibd^{-1}cT + b + i\,{}^t d^{-1}T \\ &= bd^{-1}(icT+d) + i\,{}^t d^{-1}T \end{aligned} \tag{7.49}$$

We can then apply this to $\gamma\langle iT\rangle$.

$$\begin{aligned}\gamma\langle iT\rangle &= (aiT+b)(ciT+d)^{-1} \\ &= (bd^{-1}(icT+d) + i\,{}^t d^{-1}T)(ciT+d)^{-1} \\ &= bd^{-1} + i\,{}^t d^{-1}T(ciT+d)^{-1} \\ &= bd^{-1} + iT_1 \end{aligned} \tag{7.50}$$

and we have our claim. We now want to write out the full summation form of $\theta_F^{(n)}(\gamma\langle iT\rangle)$ using the above result.

$$\theta_F^{(n)}(\gamma\langle iT\rangle) = \theta_F^{(n)}(bd^{-1} + iT_1) = \sum_M \exp(\pi i\sigma(bd^{-1} + iT_1)F[M]) \tag{7.51}$$

We now apply the division algorithm to write $M = R + lQ$ where here Q is some integral matrix and each component of the matrix remainder term R is an integer between 0 and $l-1$. We write $R \bmod l$. We now consider how to expand $F[M]$:

$$\begin{aligned} F[M] = {}^t MFM &= {}^t(R+lQ)F(R+lQ) \\ = ({}^t R + l\,{}^t Q)F(R+lQ) &= {}^t RFR + l\,{}^t QFR + l\,{}^t RFQ + l^2\,{}^t QFQ \\ = F[R] + l^2F[Q] + l({}^t QFR + {}^t RFQ) &= F[R] + 2l\,{}^t QFR + l^2F[Q] \end{aligned} \tag{7.52}$$

where the last step comes from the fact that F defines a symmetric bilinear form and thus ${}^t QFR$ is equal to ${}^t RFQ$. We can now write our theta series as

$$\begin{aligned}\theta_F^{(n)}(\gamma\langle iT\rangle) &= \sum_{M\in \mathrm{M}_{m\times n}(\mathbb{Z})} \exp(\pi i\sigma((bd^{-1} + iT_1)F[M])) \\ &= \sum_{Q\in \mathrm{M}_{m\times n}(\mathbb{Z})} \sum_{R \bmod l} \exp(\pi i\sigma((bd^{-1} + iT_1)(F[R] + 2l\,{}^t QFR + l^2F[Q]))) \end{aligned} \tag{7.53}$$

We will now expand out the brackets in the exponent to get six terms.

$$\begin{aligned}\theta_F^{(n)}(\gamma\langle iT\rangle) = \sum_{Q\in \mathrm{M}_{m\times n}(\mathbb{Z})} \sum_{R \bmod l} \exp\{&\pi i\sigma(bd^{-1}F[R]) + \pi i\sigma(2lbd^{-1}\,{}^t QFR) + \pi i\sigma(bd^{-1}l^2F[Q]) \\ &+\pi i\sigma(iT_1F[R]) + \pi i\sigma(2ilT_1\,{}^t QFR) + \pi i\sigma(il^2T_1F[Q])\}\end{aligned} \tag{7.54}$$

We now claim that $\sigma(2lbd^{-1}\,{}^tQFR)$ and $\sigma(bd^{-1}l^2F[Q])$ are both even integers. In which case, $\exp(\pi i\sigma(2lbd^{-1}\,{}^tQFR) + \pi i\sigma(bd^{-1}l^2F[Q])) = 1$ and we can discard these terms in the exponential above.

Firstly, $\sigma(bd^{-1}l^2F[Q]) = \sigma(b\cdot ld^{-1}\cdot lF[Q])$. We know that b is an integer-valued matrix. Likewise ld^{-1} is integral by the definition of l. We of course have that l is an integer and finally $F[Q]$ has integer entries and even diagonal since F had even diagonal. Therefore $b\cdot ld^{-1}\cdot lF[Q]$ is an integral matrix with even diagonal. Thus $\sigma(b\cdot ld^{-1}\cdot lF[Q])$ is even and so $\sigma(bd^{-1}l^2F[Q])$ is an even integer.

Secondly, $\sigma(2lbd^{-1}\,{}^tQFR) = 2\sigma(b\cdot ld^{-1}\cdot {}^tQFR)$ and by the same argument it is clear to see how this is even.

Therefore our theta series becomes:

$$\begin{aligned}\theta_F^{(n)}(\gamma\langle iT\rangle) = \sum_{Q\in \mathrm{M}_{m\times n}(\mathbb{Z})}\ \sum_{R \bmod l} \exp\{&\pi i\sigma(bd^{-1}F[R]) - \pi\sigma(T_1F[R]) \\ &+\pi i\sigma(2ilT_1\,{}^tQFR) + \pi i\sigma(il^2T_1F[Q])\}\end{aligned} \tag{7.55}$$

The first two terms in the exponential have no dependence on Q so we can rearrange the sum to

$$\begin{aligned}\theta_F^{(n)}(\gamma\langle iT\rangle) = &\sum_{R \bmod l} \exp\{\pi i\sigma(bd^{-1}F[R]) - \pi\sigma(T_1F[R])\} \\ \times &\sum_{Q\in \mathrm{M}_{m\times n}(\mathbb{Z})} \exp\{2\pi i\sigma(ilT_1\,{}^tQFR) + \pi i\sigma(il^2T_1F[Q])\}\end{aligned} \tag{7.56}$$

We apply the cyclic property of the trace to see that $\sigma(ilT_1\,{}^tQFR) = \sigma({}^tQ(ilFRT_1))$. We can then refer to (7.8) and see that the sum over Q is exactly the generalised theta series $\theta_F^{(n)}(il^2T_1; ilFRT_1, 0)$. So we have

$$\theta_F^{(n)}(\gamma\langle iT\rangle) = \sum_{R \bmod l} \exp\{\pi i\sigma(bd^{-1}F[R]) - \pi\sigma(T_1F[R])\}\cdot\theta_F^{(n)}(il^2T_1; ilFRT_1, 0) \tag{7.57}$$

We then apply the inversion formula (7.35) to get that

$$\begin{aligned}\theta_F^{(n)}(il^2T_1; ilFRT_1, 0) &= \det(F)^{-\frac{n}{2}}\det(-i(il^2T_1))^{-\frac{m}{2}}\theta_{F^{-1}}^{(n)}(-(il^2T_1)^{-1}; 0, -ilFRT_1) \\ &= \det(F)^{-\frac{n}{2}}\det(l^2T_1)^{-\frac{m}{2}}\theta_{F^{-1}}^{(n)}(il^{-2}(T_1)^{-1}; 0, -ilFRT_1)\end{aligned} \tag{7.58}$$

which we can plug into the expression for $\theta_F^{(n)}(\gamma\langle iT\rangle)$ to get

$$\begin{aligned}\theta_F^{(n)}(\gamma\langle iT\rangle) = \det(F)^{-\frac{n}{2}}\det(l^2T_1)^{-\frac{m}{2}}\sum_{R \bmod l} &\exp\{\pi i\sigma(bd^{-1}F[R]) - \pi\sigma(T_1F[R])\} \\ &\times\theta_{F^{-1}}^{(n)}(il^{-2}T_1^{-1}; 0, -ilFRT_1)\end{aligned} \tag{7.59}$$

We can now finally calculate the left hand side of (7.43) to be

$$\begin{aligned}\text{LHS} &= \lim_{T\to 0}\det(T)^{\frac{m}{2}}\det(ciT+d)^{-\frac{m}{2}}\theta_F^{(n)}(\gamma\langle iT\rangle)\\ &= \lim_{T\to 0}\det(T)^{\frac{m}{2}}\det(ciT+d)^{-\frac{m}{2}}\det(F)^{-\frac{n}{2}}\det(l^2T_1)^{-\frac{m}{2}}\\ &\times \sum_{R \bmod l}\exp\{\pi i\sigma(bd^{-1}F[R])-\pi\sigma(T_1F[R])\}\cdot\theta_{F^{-1}}^{(n)}(il^{-2}T_1^{-1};0,-ilFRT_1) \qquad (7.60)\end{aligned}$$

We can see that $\det(l^2T_1)^{-\frac{m}{2}} = l^{-mn}\det(T_1)^{-\frac{m}{2}}$ since T_1 is of size $n\times n$. Thus

$$\begin{aligned}\text{LHS} &= \det(F)^{-\frac{n}{2}}l^{-mn}\left(\lim_{T\to 0}\det(TT_1^{-1})^{\frac{m}{2}}\det(ciT+d)^{-\frac{m}{2}}\right)\sum_{R \bmod l}\exp\{\pi i\sigma(bd^{-1}F[R])\}\\ &\times \lim_{T\to 0}\left(\exp(-\pi\sigma(T_1F[R]))\theta_{F^{-1}}^{(n)}(il^{-2}T_1^{-1};0,-ilFRT_1)\right) \qquad (7.61)\end{aligned}$$

It is elementary to see that $\lim_{T\to 0}\det(ciT+d)^{-\frac{m}{2}} = \det(d)^{-\frac{m}{2}}$. The rest of the limits in the above are stated as fact in the paper of Andrianov and Maloletkin [5] without proof.

For the first of these we have

$$\begin{aligned}\lim_{T\to 0}\det(TT_1^{-1}) &= \lim_{T\to 0}\det(T({}^td^{-1}T(ciT+d)^{-1})^{-1})\\ &= \lim_{T\to 0}\det(T(ciT+d)T^{-1}\,{}^td) = \lim_{T\to 0}\det(T)\det(ciT+d)\det(T^{-1})\det({}^td)\\ &= \lim_{T\to 0}\det(ciT+d)\det(d) = \det(d)^2 \qquad (7.62)\end{aligned}$$

We note that as $T\to 0$, $T_1 = {}^td^{-1}T(ciT+d)^{-1}$ also tends to zero. Thus $\lim_{T\to 0}\exp(-\pi\sigma(T_1F[R])) = 1$.

Finally, we have that

$$\begin{aligned}\lim_{T\to 0}\theta_{F^{-1}}^{(n)}(il^{-2}T_1^{-1};0,-ilFRT_1) &= \lim_{T\to 0}\sum_M\exp(\pi i\sigma(il^{-2}T_1^{-1}[M+ilFRT_1]))\\ &= \sum_M\lim_{T\to 0}\exp(-\pi l^{-2}\sigma(T_1^{-1}[M+ilFRT_1])) \qquad (7.63)\end{aligned}$$

Focussing on the $M=0$ term we have

$$\lim_{T\to 0}\exp(-\pi l^{-2}\sigma(T_1^{-1}[ilFRT_1])) = \lim_{T\to 0}\exp(\pi\sigma(T_1^{-1}[FRT_1])) \qquad (7.64)$$

Since in the exponent we have two factors of T_1 and one factor of T_1^{-1}, it is clear to see that as $T\to 0$ the trace goes to 0 and thus $\lim_{T\to 0}\exp(\pi\sigma(T_1^{-1}[FRT_1])) = 1$.

So the full limit is

$$\lim_{T\to 0}\theta_{F^{-1}}^{(n)}(il^{-2}T_1^{-1};0,-ilFRT_1) = 1+\sum_{M\neq 0}\lim_{T\to 0}\exp(-\pi l^{-2}\sigma(T_1^{-1}[M+ilFRT_1])) \qquad (7.65)$$

We have that the trace term is equal to

$$\sigma(T_1^{-1}[M + ilFRT_1]) = \sigma(T_1^{-1}[M] + T_1^{-1}[ilFRT_1] + 2\,{}^tMT_1^{-1}ilFRT_1) \tag{7.66}$$

where we can group the cross terms by a similar argument to (7.52). Here, as $T \to 0$ we have $T_1^{-1} \to \infty$ (in the sense that its entries go to infinity) and so the $T_1^{-1}[M]$ term dominates and goes to $+\infty$ as T_1^{-1} is positive definite. Therefore the trace $\sigma(T_1^{-1}[M + ilFRT_1]) \to +\infty$ which gives us that $\lim_{T\to 0} \exp(-\pi l^{-2}\sigma(T_1^{-1}[M + ilFRT_1])) = 0$ for all $M \neq 0$. Hence we deduce that

$$\lim_{T\to 0} \theta^{(n)}_{F^{-1}}(il^{-2}T_1^{-1}; 0, -ilFRT_1) = 1 \tag{7.67}$$

We now have all the component limits of (7.61) and so can calculate the simplified expression of the left hand side to be

$$\begin{aligned} \text{LHS} &= \det(F)^{-\frac{n}{2}} l^{-mn} (\det(d)^2)^{\frac{m}{2}} \det(d)^{-\frac{m}{2}} \times \sum_{R \bmod l} \exp(\pi i \sigma(bd^{-1}F[R])) \\ &= l^{-mn} \det(F)^{-\frac{n}{2}} |\det(d)|^{\frac{m}{2}} \sum_{R \bmod l} \exp(\pi i \sigma(bd^{-1}F[R])) \end{aligned} \tag{7.68}$$

We can now equate this with our expression for the right hand side (7.47) to get

$$\begin{aligned} l^{-mn} \det(F)^{-\frac{n}{2}} |\det(d)|^{\frac{m}{2}} \sum_{R \bmod l} \exp(\pi i \sigma(bd^{-1}F[R])) &= \chi_F^{(n)}(\gamma) \det(F)^{-\frac{n}{2}} \\ l^{-mn} |\det(d)|^{\frac{m}{2}} \sum_{R \bmod l} \exp(\pi i \sigma(bd^{-1}F[R])) &= \chi_F^{(n)}(\gamma) \end{aligned} \tag{7.69}$$

The above equation is exactly what we wanted to prove. □

7.4.3 Character Theorem Part 2: Computation of the Gauss Sum

We will now compute the Gauss sum in Proposition 78, and thus we will be able to find the character.

Firstly, we will note that in the case of degree $n = 1$ (i.e. for classical theta series) the Gauss sum is calculated in the book of Eichler [11] and thus the following theorem will be taken as given.

Theorem 79. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)$, $m = 2k$.

$$\chi_F^{(1)}(\gamma) = (\text{sign}(d))^k \left(\frac{(-1)^k \det(F)}{|d|} \right) \tag{7.70}$$

Where here $\left(\frac{\cdot}{n}\right)$ denotes the generalised Legendre symbol (i.e. the Legendre symbol extended to allow non-prime denominators).

The calculation of the character for degree $n \geq 2$ will rely on reducing the calculation to the $n = 1$ case. We will also need some group theory which we will now discuss.

Definition 80. *Let K be the subgroup of $\Gamma_0^{(n)}(N)$ generated by matrices of the following form:*

$$\hat{u} = \begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix} \; ; \; t = \begin{pmatrix} \mathbb{1} & s_1 \\ 0 & \mathbb{1} \end{pmatrix} \; ; \; w = \begin{pmatrix} \mathbb{1} & 0 \\ Ns_2 & \mathbb{1} \end{pmatrix} \tag{7.71}$$

where $u \in U_n^+$ and s_1, s_2 are $n \times n$ integral symmetric matrices.

Definition 81. *We say that two matrices $\gamma, \gamma' \in \Gamma_0^{(n)}(N)$ are K-equivalent if $\gamma' = A\gamma B$ for some $A, B \in K$. If γ and γ' are K-equivalent we denote this by $\gamma \sim \gamma'$.*

Andrianov and Maloletkin [5] state the following lemma as a fact without explanation. We shall make sure to justify it properly

Lemma 82. *Let $\gamma \sim \gamma'$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then $\det(d) \equiv \det(d') \bmod N$.*

Proof. We will check what happens to the bottom right block of modular matrices γ under left and right multiplication by the generators $\hat{u}, t, w$.

$$\begin{aligned}
\hat{u}\gamma = \begin{pmatrix} * & * \\ * & {}^t u^{-1} d \end{pmatrix} \quad &\implies \quad \det({}^t u^{-1} d) = \det({}^t u^{-1})\det(d) = \det(d) \\
\gamma\hat{u} = \begin{pmatrix} * & * \\ * & d\,{}^t u^{-1} \end{pmatrix} \quad &\implies \quad \det(d\,{}^t u^{-1}) = \det(d)\det({}^t u^{-1}) = \det(d) \\
t\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \quad &\implies \quad \det(d) \text{ invariant} \\
\gamma t = \begin{pmatrix} * & * \\ * & cs_1 + d \end{pmatrix} \quad &\implies \quad \det(cs_1 + d) \equiv \det(d) \bmod N \text{ since } c \equiv 0 \bmod N \\
w\gamma = \begin{pmatrix} * & * \\ * & Ns_2 b + d \end{pmatrix} \quad &\implies \quad \det(Ns_2 b + d) \equiv \det(d) \bmod N \\
\gamma w = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \quad &\implies \quad \det(d) \text{ invariant}
\end{aligned} \tag{7.72}$$

□

We will also require the following result stated as lemma 5 in the paper of Andrianov and Maloletkin [5].

Proposition 83. *For all $\gamma \in \Gamma_0^{(n)}(N)$, there exists a $\gamma_0 \in \Gamma_0^{(n)}(N)$ such that $\gamma \sim \gamma_0$ with*

$$\gamma_0 = \left(\begin{array}{ccccc|ccccc}
1 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\
0 & \cdot & \cdot & \cdot & a_0 & 0 & \cdot & \cdot & \cdot & b_0 \\
\hline
0 & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & 0 \\
\cdot & 0 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
0 & \cdot & \cdot & \cdot & c_0 & 0 & \cdot & \cdot & \cdot & d_0
\end{array}\right) \tag{7.73}$$

We will omit the full proof of this since the steps are well documented in the paper of Andrianov and Maloletkin [5], and there is not much to add.

The proof strategy boils down to showing the weaker statement, that any $\gamma \in \Gamma_0^{(n)}(N)$ is K-equivalent to a matrix of the form

$$\gamma' = \left(\begin{array}{cccc|cccc} 1 & \dots & 0 & & 0 & \dots & 0 & \\ \vdots & a' & & & \vdots & b' & & \\ 0 & & & & 0 & & & \\ \hline 0 & \dots & 0 & & 1 & \dots & 0 & \\ \vdots & c' & & & \vdots & d' & & \\ 0 & & & & 0 & & & \end{array}\right) \tag{7.74}$$

where $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0^{(n-1)}(N)$. Andrianov and Maloletkin [5] then use induction on the degree n to prove the proposition.

We are now finally in a position to be able to calculate the group character for general $n \geq 2, N \geq 2$ and $m = 2k$.

Theorem 84. *If $N > 1$, $m = 2k$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have that the group character $\chi_F^{(n)}(\gamma) = \chi_F(\det(d))$ where χ_F is a real Dirichlet character defined to be*

$$\chi_F(v) := (\operatorname{sign}(v))^k \left(\frac{(-1)^k \det(F)}{|v|}\right) \tag{7.75}$$

where $v \in \mathbb{Z}$ and $\left(\frac{\cdot}{n}\right)$ is the generalised Legendre symbol.

Proof. We begin by recalling the formula for the character given in Proposition 78.

$$\chi_F^{(n)}(\gamma) = |\det(d)^{\frac{m}{2}}|\; l^{-mn} \sum_{R \bmod l} \exp(\pi i \sigma(bd^{-1}F[R])) \tag{7.76}$$

We will use this to show that $\chi_F^{(n)}(\gamma) = 1$ for all $\gamma \in K$. Andrianov and Maloletkin [5] state this as a triviality but we will check that it makese sense. It suffices to show that this is true for the generators.

For the three categories of generators $\hat{u}$,t and w we have that the bottom right corner d is unimodular with determinant $+1$, thus d^{-1} has integer entries. Recall that l was an integer such that ld^{-1} was an integral matrix, so we can just pick $l = 1$.

Therefore the condition R mod l just reduces to R mod 1 which means the sum is over a single term where R is just the zero matrix. Thus for $\gamma = \hat{u}$, $\gamma = t$ or $\gamma = w$ we have

$$\chi_F^{(n)}(\gamma) = |\det(d)^{\frac{m}{2}}| \cdot l^{-mn} \cdot \exp(\pi i \sigma(bd^{-1}F[0])) = 1^{\frac{m}{2}} \cdot 1^{-mn} \cdot 1 = 1 \tag{7.77}$$

The Group character is given in terms of a Dirichlet character. Dirichlet characters are multiplicative, and the character is trivial on the K subgroup. Therefore for general γ it suffices to calculate the character for a K-equivalent matrix of the form γ_0 given in Proposition 83. Andrianov and Maloletkin [5] immediately jump to the conclusion from here but there are still some important steps to take which we shall detail in full.

For γ_0, the bottom right corner d has determinant d_0, and it is clear to see that the matrix $d_0 d^{-1}$ has integer entries - thus we pick $l = d_0$. Hence for all $\gamma \in \Gamma_0^{(n)}(N)$:

$$\chi_F^{(n)}(\gamma) = \chi_F^{(n)}(\gamma_0) = d_0^{\frac{m}{2}} \cdot d_0^{-mn} \cdot \sum_{R \bmod d_0} \exp(\pi i \sigma(bd^{-1}F[R])) \tag{7.78}$$

We will investigate the trace term in the exponent. We see that

$$bd^{-1} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & b_0 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & d_0^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & b_0 d_0^{-1} \end{pmatrix} \tag{7.79}$$

which is a matrix of zeros with the exception of $b_0 d_0^{-1}$ in the nn-th entry. Therefore the matrix $bd^{-1}F[R]$ will have all zeros in every row except the n-th row:

$$\begin{aligned} bd^{-1}F[R] &= b_0 d_0^{-1} \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 \\ \hline \multicolumn{5}{c}{\text{Last row of } F[R]} \end{pmatrix} \\ \implies \sigma(bd^{-1}F[R]) &= b_0 d_0^{-1}[F[R]]_{nn} = b_0 d_0^{-1}\, {}^t R_n F R_n \end{aligned} \tag{7.80}$$

where here R_n denotes the n-th column of R. We should therefore be able to reduce the sum from being over matrices R mod d_0 to over vectors R_n mod d_0 with $R_n \in \mathbb{Z}^m$.

However - we must be careful here since every vector R_n will correspond to multiple matrices R all sharing the same last column. Indeed, each R_n corresponds to $d_0^{m(n-1)}$ matrices R because there are $m(n-1)$ 'free variables' not fixed by the choice of R_n, each of which can take a value between $0, ..., d_0 - 1$. Thus we can take out a factor of $d_0^{m(n-1)}$ and reduce the calculation to:

$$\chi_F^{(n)}(\gamma) = d_0^{\frac{m}{2}} \cdot d_0^{-mn} \cdot d_0^{m(n-1)} \cdot \sum_{R_n \bmod d_0} \exp(\pi i b_0 d_0^{-1} F[R_n]) = d_0^{-\frac{m}{2}} \sum_{R_n \bmod d_0} \exp(\pi i b_0 d_0^{-1} F[R_n]) \tag{7.81}$$

By Proposition 78 with $n = 1$ we have

$$\chi_F^{(1)} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = d_0^{-\frac{m}{2}} \sum_{R \in \mathbb{Z}^m \bmod d_0} \exp(\pi i \sigma(b_0 d_0^{-1} F[R])) \tag{7.82}$$

which is clearly equal to $\chi_F^{(n)}(\gamma)$ after some re-labelling. We know a formula for calculating the character in the $n = 1$ case courtesy of Theorem 79 (proven in the book of Eichler [11]).

Therefore we deduce:

$$\chi_F^{(n)}(\gamma) = \chi_F^{(1)} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = (\text{sign}(d_0))^k \left(\frac{(-1)^k \det(F)}{|d_0|} \right) = \chi_F(d_0) \tag{7.83}$$

since $d_0 \equiv \det(d) \bmod N$ we can deduce $\chi_F(d_0) = \chi_F(\det(d))$ because χ_F is a Dirichlet character modulo N. Thus we have $\chi_F^{(n)}(\gamma) = \chi_F(\det(d))$ as required. □

Chapter 8

Discussion and further considerations

We recall that our ultimate motivation for studying theta series was to find explicit or asymptotic formulae for the representation numbers $r_F(G)$.

We recall from the study of classical theta series [12] the following theorem:

Theorem 85. *Let $F \in \mathrm{GL}_{2k}(\mathbb{Z})$ be positive definite, denote the k-th Bernoulli Number by B_k. Then for the representation number for $g \in \mathbb{N}$ by F we have:*

$$r_F(g) = -\frac{2k}{B_k}\sigma_{k-1}(g) + \mathcal{O}(g^{k/2}) \tag{8.1}$$

where $\sigma_s(g) = \sum_{d|g} d^s$.

The proof involves knowing that $\theta_F^{(1)} \in \mathcal{M}_k^{(1)}(1, \mathrm{Id})$. i.e. the theta series associated with F is a modular form of weight k for the full modular group with trivial character.

By this fact we are able to write the theta series in terms of an Eisenstein series component and a cuspidal component. The representation numbers can then be written in terms of the Fourier coefficients of the Eisensten series (which gives the explicit part) and the cuspidal component (which gives the asymptotic part).

In general for Siegel modular forms of degree $n \geq 2$ this type of reasoning takes a lot more effort. There exist analogues of Eisenstein series and cusp forms [17] for Siegel modular forms, however they are more difficult to pin down.

Per page 46 the paper of Böcherer [7] there is a theorem of Siegel and Witt which relates Siegel theta series and Eisenstein series (of 'Siegel type'). The paper is written in German, but a translation of 'Satz 21' is written below:

Theorem 86. *Let $F_1, ..., F_h$ be a full set of $\mathrm{GL}_m(\mathbb{Z})$ representatives of symmetric, positive definite, unimodular matrices with even diagonal. Define $\epsilon(F_\mu) := \#\{M \in \mathrm{M}_m(\mathbb{Z}) : {}^tMF_\mu M = F_\mu\}$.*

$$\sum_{\mu=1}^{h} \frac{\theta_{F_\mu}^{(n)}(z)}{\epsilon(F_\mu)} = \sum_{\mu=1}^{h} \frac{\Psi_n^{m/2}(z)}{\epsilon(F_\mu)} \tag{8.2}$$

where $\Psi_n^{m/2}(z)$ denotes the Eisenstein series of Siegel type - which is a degree n, weight $m/2$ Siegel modular form.

Such unimodular F_μ exist if and only if $8|m$ [19], which is exactly the same as the result proven for classical theta series [12] because the level of the theta series depends only on the matrix F and not the degree n. Comparing Fourier coefficients in (8.2) allows us to generate expressions for the representation numbers.

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Appendix A

Glossary of Notation

Here is a reference table of all the notation and conventions used throughout this report

Notation	Meaning	Notes
$\mathrm{M}_{m\times n}(R)$	The space of $m \times n$ matrices with entries in the ring R	$\mathrm{M}_n(R) := \mathrm{M}_{n\times n}(R)$
${}^t a$	The transpose of a matrix a.	
$a[b]$	${}^t bab$	$a \in \mathrm{M}_n(\mathbb{C}), b \in \mathrm{M}_{m\times n}(\mathbb{C})$
$a\{b\}$	${}^t\bar{b}ab$	a and b as above
$y > 0$	y is positive definite, where y is some square matrix	
$y \geq 0$	y is positive semi-definite, with y as above	
$A > B$	The Loewner ordering: $A - B$ is positive definite	$A, B \in \mathrm{M_n}(\mathbb{R})$
$\mathbb{H}_n$	Siegel's half-space of degree n	
$\mathbb{H}_n^{\mathrm{Re}}, \mathbb{H}_n^{\mathrm{Im}}$	The real and imaginary subsets of $\mathbb{H}_n$	
$\mathrm{Sp}(n, \mathbb{R})$	The symplectic group of degree n	
j	$j = \begin{pmatrix} 0_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0_{n\times n} \end{pmatrix}$	
$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	Matrices in the symplectic group	
$\gamma\langle z\rangle$	The group action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathbb{H}_n$	
$\sigma(m)$	The trace of a matrix m	
$B_r(w)$	The open ball of radius r centred at w	w in some metric space
Γ_n	Siegel's modular group of degree n, $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$	
U_n	The unimodular group $\mathrm{GL}(n, \mathbb{Z})$	
U_n^+	The subgroup of U_n of matrices with positive determinant	
$\hat{U}_n$	The unimodular subgroup of Γ_n, $\left\{\begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} : u \in U_n\right\}$	
T	The translation subgroup of Γ_n, $\left\{\begin{pmatrix} \mathbb{1} & s \\ 0 & \mathbb{1} \end{pmatrix} : s \in \mathrm{M}_n(\mathbb{Z}),\ {}^t s = s\right\}$	
A	The subgroup of Γ_n of integral modular substitutions, $\left\{\begin{pmatrix} u & s\,{}^t u^{-1} \\ 0 & {}^t u^{-1} \end{pmatrix} : u \in U_n\ ;\ s \in \mathrm{M}_n(\mathbb{Z}),\ {}^t s = s\right\}$	
$M \equiv 0 \bmod N$	Every entry of M is a multiple of N	$M \in \mathrm{M}_{m\times n}(\mathbb{Z})$

Notation	Meaning	Notes
$\Gamma_0^{(n)}(N)$	The Hecke congruence subgroup of level N $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Z}) : c \equiv 0 \bmod N\right\}$	
$[c,d]$	The set of bottom halves of modular matrices associated to (c,d)	
$[\gamma]$	The set of $\gamma' \in \mathrm{Sp}(n,\mathbb{Z})$ with bottom halves associated to the bottom half of γ	
$\mathcal{O}(p)$	$\{\gamma\langle p\rangle : \gamma \in \Gamma\}$ The group orbit of $p \in P$ under the action of a group Γ.	
R_n	Minkowski's reduced domain	
$j(\gamma, z)$	$\det(cz+d)$, The factor of automorphy	
y^D	$y^D := \mathrm{diag}(y_{1,1}, y_{2,2}, ..., y_{n,n})$	$y \in \mathrm{M}_n(R)$
$[A]_{ij}$	The i,j-th entry of A. Sometimes write A_{ij} or $A_{i,j}$	$A \in \mathrm{M}_{m\times n}(R)$
$\mathcal{F}_n$	Siegel's Fundamental Domain	
χ	A Dirichlet character	
$\chi^{(n)}$	A character on the group $\Gamma_0^{(n)}(N)$ defined in terms of a Dirichlet character modulo N, $\chi^{(n)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \chi(\det(d))$	
$\mathcal{M}_k^{(n)}(N,\chi)$	The vector space of Siegel modular forms of degree n, weight k, level N and character χ	
$\theta_F^{(n)}(z)$	The theta series of degree n associated with a matrix F	
$r_F(G)$	$\#\{M \in \mathrm{M}_{m\times n}(\mathbb{Z}) : {}^tMFM = G\}$ The representation number of G by F	
$\theta_F^{(n)}(z; X, Y)$	The generalised theta series	
$\vartheta_n(\tilde{z})$	The theta function of degree n	
$\Theta^{(n)}$	The theta group $\{\gamma \in \mathrm{Sp}(n,\mathbb{Z}) : {}^tca \text{ and } {}^tbd \text{ have even diagonal}\}$	
ϕ	A map from $\mathbb{H}_n$ to $\mathbb{H}_{mn}$ defined by $\phi(z) := F \otimes z$	
φ	A map from $\mathrm{M}_{m\times n}(\mathbb{C})$ to $\mathbb{C}^{mn}$ defined by $\varphi((M_1,\ldots,M_n)) := \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}$ where M_i are the columns of M	
Φ	A map from $\Gamma_0^{(n)}(N)$ to $\Theta^{(mn)}$ defined by $\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} \mathbb{1}_m \otimes a & F \otimes b \\ F^{-1} \otimes c & \mathbb{1}_m \otimes d \end{pmatrix}$	
$\vartheta_n(\tilde{z}; \tilde{\mathbf{X}}, \tilde{\mathbf{Y}})$	The generalised theta function	
K	The subgroup of $\Gamma_0^{(n)}(N)$ generated by matrices of the form $\hat{u} \in \hat{U}_n^+\ ;\ t \in T\ ;\ w = \begin{pmatrix} \mathbb{1} & 0 \\ Ns_2 & \mathbb{1} \end{pmatrix}$ where s_2 is an integral symmetric matrix.	
$\gamma \sim \gamma'$	γ and γ' are K-equivalent	
$\sigma_s(g)$	The divisor function $\sum_{d\mid g} d^s$	
$\Psi_n^k(z)$	The Eisenstein Series of Siegel type of weight k and degree n	

Appendix B

Tensor Products

Throughout Chapter 7 on theta series we use a number of tensor product identities. Many of these can be found in the Durham University course on Quantum Computing [24] (Although with an additional focus on complex matrices). All of these properties are stated in the paper of Andrianov and Maloletkin without proof [5]. We will state them here, and provide a proof for a couple of the less obvious identities.

Theorem 87. *Let $A \in \mathrm{M}_{m\times m}(\mathbb{R}), B \in \mathrm{M}_{n\times n}(\mathbb{R})$, with $[A]_{ij} =: a_{ij}$ and $[B]_{ij} =: b_{ij}$.*
The tensor product of A and B is defined as

$$A \otimes B := \begin{pmatrix} Ab_{11} & \cdot & \cdot & \cdot & Ab_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ Ab_{n1} & \cdot & \cdot & \cdot & Ab_{nn} \end{pmatrix} \in \mathrm{M}_{mn\times mn}(\mathbb{R}) \tag{B.1}$$

and the following properties hold:

- $(A \otimes B)(A' \otimes B') = AA' \otimes BB'$
- *If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$*
- $\det(A \otimes B) = \det(A)^n \cdot \det(B)^m$
- ${}^t(A \otimes B) = {}^tA \otimes {}^tB$
- *If A and B are positive definite, then so is $A \otimes B$.*

We shall prove the transpose identity and the positive definite identity.

Proof. First the transpose identity: By the definition of transpose we have

$${}^t(A \otimes B) = \begin{pmatrix} {}^tAb_{11} & \cdot & \cdot & \cdot & {}^tAb_{n1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ {}^tAb_{1n} & \cdot & \cdot & \cdot & {}^tAb_{nn} \end{pmatrix} = {}^tA \otimes {}^tB \tag{B.2}$$

We now show the positive definite identity using the first method given in [3]. We recall that a matrix is positive definite if and only if all of its eigenvalues are positive.

Take a complete eigenbasis $\{\mathbf{v}_i\}_{1\leq i\leq m}$ for A with eigenvalues λ_i respectively. Likewise we take a complete eigenbasis $\{\mathbf{w}_j\}_{1\leq j\leq n}$ for B with eigenvalues μ_j respectively.

Then

$$(A\otimes B)(\mathbf{v}_i\otimes\mathbf{w}_j)=A\mathbf{v}_i\otimes B\mathbf{w}_j=\lambda_i\mathbf{v}_i\otimes\mu_j\mathbf{w}_j=\lambda_i\mu_j(\mathbf{v}_i\otimes\mathbf{w}_j) \tag{B.3}$$

So $\lambda_i\mu_j$ is an eigenvalue for $A\otimes B$. Indeed, since A and B are both diagonalisable, they have m and n eigenvalues respectively, counting multiplicity. Therefore we know the set $\{\lambda_i\mu_j\}$ contains mn eigenvalues, counting multiplicity, and thus this is a complete set of eiganvalues for $A\otimes B$. Since $\lambda_i>0$ for all i and $\mu_j>0$ for all j we therefore deduce that $\lambda_i\mu_j>0$ for all i,j and so $A\otimes B$ is positive definite.

□

Appendix C

Miscellaneous proofs and expository discussions

In this Appendix chapter, we provide a number of additional results which are tangentially relevant to the content of the report. Each section is titled after the chapter in the main body which it is relevant to.

C.1 The Symplectic Group

Here we include a quick proof that the Symplectic Group given in Definition 8 is well-defined, as well as a small discussion about the relevance of this group.

C.1.1 The Symplectic Group is a group

We will quickly check that the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ is indeed a group. Firstly, the identity $\mathbb{1}_{2n}$ is in the group because $j[\mathbb{1}_{2n}] = j$. We also have that the inverse of an element γ is $\gamma^{-1} = -j\,{}^t\gamma j \in \mathrm{Sp}(n, \mathbb{R})$.

Indeed:

$$\gamma\gamma^{-1} = -\gamma j\,{}^t\gamma j = -j^2 = \mathbb{1} \tag{C.1}$$

Finally we must check closure. We take $\gamma_1, \gamma_2 \in \mathrm{Sp}(n, \mathbb{R})$.

$j[\gamma_1\gamma_2] = (j[\gamma_1])[\gamma_2] = j[\gamma_2] = j$ so $\gamma_1\gamma_2 \in \mathrm{Sp}(n, \mathbb{R})$ and so $\mathrm{Sp}(n, \mathbb{R})$ is a group.

C.1.2 Justification

Klingen's book [17] does not give much justification for definition 8, and to the average reader the condition $j[m] = j$ may seem arbitrary and random. Maass [22] has some information on automorphism groups of bilinear forms in chapter 2 of his book, but the level of detail is unnecessary for our discussions.

We shall explore why this definition is meaningful.

Consider the vector space $V = \mathbb{R}^{2n}$ and a bilinear form on this vector space,

$$\begin{aligned} \phi &: V \times V \to \mathbb{R} \\ \phi(\mathbf{v_1}, \mathbf{v_2}) &= {}^t\mathbf{v_1} j \mathbf{v_2} \end{aligned} \tag{C.2}$$

where $j \in M_{2n}(\mathbb{R})$ is the matrix defined in Definition 8.

Now consider an arbitrary $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ and we investigate what happens when we transform the inputs of ϕ by this matrix.

$$\begin{aligned}\phi(\gamma\mathbf{v_1}, \gamma\mathbf{v_2}) &= {}^t(\gamma\mathbf{v_1})j(\gamma\mathbf{v_2}) = {}^t\mathbf{v_1}\,{}^t\gamma j\gamma\mathbf{v_2} = {}^t\mathbf{v_1}j[\gamma]\mathbf{v_2} \\ &= \mathbf{v_1}j\mathbf{v_2} = \phi(\mathbf{v_1}, \mathbf{v_2})\end{aligned} \tag{C.3}$$

So for all $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ we have that $\phi(\gamma\mathbf{v_1}, \gamma\mathbf{v_2}) = \phi(\mathbf{v_1}, \mathbf{v_2})$ so the bilinear form is preserved under transformation by matrices in the symplectic group. We should also check that this relationship is an equivalence; that all matrices preserving the bilinear form indeed lie in the symplectic group.

Suppose we have an arbitrary $\gamma \in \mathrm{GL}(2n, \mathbb{R})$ such that $\phi(\gamma\mathbf{v_1}, \gamma\mathbf{v_2}) = \phi(\mathbf{v_1}, \mathbf{v_2})$ for all $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^{2n}$. Then we have

$${}^t(\gamma\mathbf{v_1})j(\gamma\mathbf{v_2}) = \mathbf{v_1}j\mathbf{v_2} \tag{C.4}$$

and thus

$${}^t\mathbf{v_1}\,{}^t\gamma j\gamma\mathbf{v_2} = \mathbf{v_1}j\mathbf{v_2} \tag{C.5}$$

Since this is true for arbitrary $\mathbf{v_1}$ and $\mathbf{v_2}$ it must therefore be true that ${}^t\gamma j\gamma = j$ and thus $\gamma \in \mathrm{Sp}(n, \mathbb{R})$.

Why is this bilinear form in particular interesting? Note that ${}^tj = -j$, so we can deduce that

$$\phi(\mathbf{v_1}, \mathbf{v_2}) = {}^t\mathbf{v_1}j\mathbf{v_2} = -\,{}^t\mathbf{v_1}\,{}^tj\mathbf{v_2} = -\,{}^t({}^t\mathbf{v_2}j\mathbf{v_1}) = -\phi(\mathbf{v_2}, \mathbf{v_1}) \tag{C.6}$$

So the matrix j defines an anti-symmetric bilinear form on $\mathbb{R}^{2n}$.

In fact, one can show that every anti-symmetric bilinear form can be associated with the matrix j after a change of basis.

C.2 The Group Action

Here we add some more detailed discussion about properties of the group action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathbb{H}_n$.

C.2.1 Biholomorphisms of $\mathbb{H}_n$

As per Klingen [17], we note that the mappings $z \mapsto \gamma\langle z\rangle$ are bijective for fixed γ. This is because there is an obvious inverse mapping $z \mapsto \gamma^{-1}\langle z\rangle$.

We call these mappings 'Symplectic mappings'. They are rational functions of z so are therefore holomorphic. Since the inverse map is performed using γ^{-1} we can also conclude that they are biholomorphic.

We construct a group homomorphism:

$$\begin{aligned}\varphi \;:\; \mathrm{Sp}(n, \mathbb{R}) &\;\to\; \mathrm{Bihol}(\mathbb{H}_n) \\ \gamma &\;\mapsto\; \gamma\langle\cdot\rangle\end{aligned} \tag{C.7}$$

Lemma 88. *The kernal of this homomorphism is* $\mathrm{Ker}(\varphi) = \{\pm\mathbb{1}\}$

The methods of proof in Klingen [17] and Maass's [22] books are essentially the same - but Maass explains the steps in extra detail.

Proof. The kernal consists of all $\gamma \in \mathrm{Sp}(n, \mathbb{R})$ such that $\gamma\langle z\rangle = z$ for all $z \in \mathbb{H}_n$. This identity is equivalent to $az + b = z(cz + d)$ by the definition of the group action.

Expanding the brackets and rearranging gives us

$$zcz + zd - az - b = 0 \ \forall z \in \mathbb{H}_n \tag{C.8}$$

Since this is true for all z, we know that the 'degree n' terms in the above relation can be treated separately (this is easier to see when working in a polynomial ring - but the same logic holds for non-commutative scenarios). Hence we get

$$\begin{aligned} -b &= 0 \\ zd - az &= 0 \\ zcz &= 0 \end{aligned} \tag{C.9}$$

Clearly this shows that $b = c = 0$ and $a = d$. The latter can be deduced by considering $z = i\mathbb{1} \in \mathbb{H}_n$. Klingen and Maass then both assert that this also implies that $a = d = \lambda\mathbb{1}$ for some $\lambda \in \mathbb{R}$. This fact is actually highly non-trivial and a proof follows (with some adjustments) from results in Newman's paper on commuting matrices [23].

Therefore we have $m = \lambda\mathbb{1}$, and by $\det(\gamma) = 1$, we conclude that $\lambda = \pm 1$ as required.

□

It can also be shown that the isomorphism φ is surjective, so by the first isomorphism theorem we have that the group of biholomorphisms of Siegel's half-space $\mathbb{H}_n$ is isomorphic to the Symplectic group of degree n over $\mathbb{R}$, quotient plus or minus the identity:

$$\mathrm{Bihol}(\mathbb{H}_n) \cong \mathrm{Sp}(n, \mathbb{R})/\{\pm\mathbb{1}\} \tag{C.10}$$

The classical case

We once again consider reducing this result to the $n = 1$ case. Indeed (C.10) becomes

$$\mathrm{Bihol}(\mathbb{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm\mathbb{1}\} \tag{C.11}$$

as we would expect.

C.3 Discrete Subgroups

C.3.1 Discrete subgroups acting discontinuously

Here we complete the proof that discrete subgroups of $\mathrm{Sp}(n, \mathbb{R})$ act on $\mathbb{H}_n$ discontinuously. The proof can also be found in [17] in slightly less detail.

Proof. We want to show that if G is discrete it must act discontinuously.

We prove via contrapositive, so we shall let the action of G be non-discontinuous. So there exists a point $z \in \mathbb{H}_n$ such that the group orbit $\{\gamma\langle z\rangle : \gamma \in G\}$ contains an accumulation point z^*.

Thus we can define a convergent sequence $\{z_v\}_{v\in\mathbb{N}}$ with $z_v \in \{\gamma\langle z\rangle : \gamma \in G\}$ with mutually distinct elements which has its limit at this accumulation point. Since each z_v is in the group orbit of z there exists a sequence of mutually distinct elements $\gamma_v \in G$ such that $z_v = \gamma_v\langle z\rangle$. We write $\gamma_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$ in $n \times n$ blocks.

Let $z = x + iy$ and $z_v = x_v + iy_v$ and recall equation (3.25):

$$y_v = y\{\overline{q}^{-1}\} \tag{C.12}$$

where $q = (c_v z + d_v)$.

We now rearrange this a little:

$$\begin{aligned} y_v &= {}^t q^{-1} y \overline{q}^{-1} \\ \implies y_v^{-1} &= ({}^t q^{-1} y \overline{q}^{-1})^{-1} \\ y_v^{-1} &= \overline{q} y^{-1}\, {}^t q \\ y_v^{-1} &= y^{-1}\{{}^t q\} = y^{-1}\{{}^t(c_v z + d_v)\} \end{aligned} \tag{C.13}$$

y_v and y are both positive definite, so their inverses y_v^{-1} and y^{-1} are in a sense 'bounded' according to Klingen [17]. (Note that y_v^{-1} is bounded with respect to v)

By (C.13) we therefore see that $(c_v z + d_v)$ is also bounded - and in fact it is possible to deduce that c_v and d_v are bounded by decomposing everything into real and imaginary parts. [17]

Furthermore, we recall that $a_v z + b_v = z_v(c_v z + d_v)$. This equation further allows us to say that a_v and b_v are bounded.

All 4 of the $n \times n$ blocks of γ_v are bounded, and so we see that the γ_v are bounded as well. Hence we have a bounded sequence $\{\gamma_v\}_{v\in\mathbb{N}}$, and therefore by the Bolzano-Weierstrass Theorem there is a convergent subsequence - and hence the group G is non-discrete.

Therefore if G is discrete, it must act discontinuously on $\mathbb{H}_n$. □

C.3.2 The Unimodular Group

Lemma 89.

$$\hat{U}_n := \left\{ \begin{pmatrix} u & 0 \\ 0 & {}^t u^{-1} \end{pmatrix} : u \in U_n \right\} \tag{C.14}$$

is a subgroup of $\mathrm{Sp}(n, \mathbb{Z})$ *isomorphic to* U_n.

Proof. Firstly we must check that $\hat{U}_n$ is indeed a group. We have that $\mathbb{1}_{2n} \in \hat{U}_n$ by setting $u = \mathbb{1}_n$. We also have a natural inverse:

$$\gamma^{-1} = \begin{pmatrix} u^{-1} & 0 \\ 0 & {}^t u \end{pmatrix} \tag{C.15}$$

for all $\gamma \in \hat{U}_n$. Closure of $\hat{U}_n$ can be seen via a quick calculation for arbitrary $a, b \in \mathrm{GL}(n, \mathbb{Z})$:

$$\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & {}^t a^{-1}\, {}^t b^{-1} \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & {}^t (ab)^{-1} \end{pmatrix} \tag{C.16}$$

So $\hat{U}_n$ is a group. Now we should check that it is a subgroup of $\mathrm{Sp}(n, \mathbb{Z})$. It suffices to check that all $\gamma \in \hat{U}_n$ satisfy $j[\gamma] = j$ which is the defining property of the Symplectic Group.

$$\begin{aligned} j[\gamma] &= {}^t\gamma j\gamma = \begin{pmatrix} {}^tu & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & {}^tu \\ -u^{-1} & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = j \end{aligned} \tag{C.17}$$

Finally, we need to show that $\hat{U}_n \cong \mathrm{GL}(n, \mathbb{Z})$. We construct the canonical isomorphism:

$$\begin{aligned} \varphi : \mathrm{GL}(n, \mathbb{Z}) &\to \hat{U}_n \\ u &\mapsto \begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix} \end{aligned} \tag{C.18}$$

It is fairly quick to check that this is indeed a homomorphism, and has inverse:

$$\begin{aligned} \varphi^{-1} : \hat{U}_n &\to \mathrm{GL}(n, \mathbb{Z}) \\ \begin{pmatrix} u & 0 \\ 0 & {}^tu^{-1} \end{pmatrix} &\mapsto u \end{aligned} \tag{C.19}$$

So we have shown:

- $\hat{U}_n$ is a group
- $\hat{U}_n$ is a subgroup of $\mathrm{Sp}(n, \mathbb{Z})$
- $\hat{U}_n \cong \mathrm{GL}(n, \mathbb{Z})$

as required. □

C.4 The Fundamental Domain

C.4.1 A Result on positive definite matrices

In the proof of Proposition 41 we claimed the following lemma, which we will prove here.

Lemma 90. *Let A,B be $n \times n$ positive definite matrices such that $A - B$ is positive definite and B is diagonal. Then* $\det(A) > \det(B)$.

Proof. If A and B are both diagonal, then this would follow immediately, since we would have $A_i > B_i$ for all i where A_i, B_i denote the diagonal entries of A and B respectively.

We will reduce this in general to the case where both A and B are diagonal.

Since B is positive definite, there exists a 'square root' matrix $B^{\frac{1}{2}}$ which is also positive definite, such that $B^{\frac{1}{2}}B^{\frac{1}{2}} = B$.

The matrix $(B^{\frac{1}{2}})^{-1}A(B^{\frac{1}{2}})^{-1}$ is symmetric, so it can be diagonalised by an orthogonal matrix P. In other words, $D = {}^tP(B^{\frac{1}{2}})^{-1}A(B^{\frac{1}{2}})^{-1}P$ is a diagonal matrix, and ${}^tPP = \mathbb{1}$.

Now let $S = (B^{\frac{1}{2}})^{-1}P$. We therefore have

$$\begin{aligned} {}^tSBS &= \mathbb{1} \\ {}^tSAS &= D \end{aligned} \tag{C.20}$$

Note that these are not 'true' diagonalisations, since in general S is not orthogonal - so the entries of the diagonal matrices in the above may not coincide with the eigenvalues of A and B.

From this we get $\det(A)\det(S)^2 = \det(D)$ and $\det(B)\det(S)^2 = 1$. Thus

$$\frac{\det(A)}{\det(B)} = \det(D) \tag{C.21}$$

We also note that $A - B > 0 \implies {}^tS(A-B)S > 0 \implies {}^tSAS - {}^tSBS > 0 \implies D - \mathbb{1} > 0$. Since D is diagonal, it is therefore clear to see that the diagonal entries $D_i > 1$ for all i and thus $\det(D) > 1$.

We hence deduce from (C.21) that $\det(A) > \det(B)$. □

C.5 Siegel Modular Forms

C.5.1 The product of two positive definite matrices

We recall that in chapters 6 and 7 we applied a result of Coope [9]. We prove this here

Lemma 91. *Take A,B positive definite $n \times n$ matrices. Then $\sigma(AB) > 0$. If A and B are positive semi-definite then we have $\sigma(AB) \geq 0$.*

Proof. We will quickly justify this for the semi-definite case as the positive definite case is identical: Since A is positive semi-definite, there exists a square root matrix $A^{\frac{1}{2}}$ which is also positive semi-definite such that $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$. By the cyclic property of the trace we thus have $\sigma(AB) = \sigma(A^{\frac{1}{2}}BA^{\frac{1}{2}})$.

Since $A^{\frac{1}{2}}$ is symmetric we have $A^{\frac{1}{2}}BA^{\frac{1}{2}} = {}^tA^{\frac{1}{2}}BA^{\frac{1}{2}}$. Then we check if this matrix is positive semi-definite. We take $\mathbf{v} \in \mathbb{R}^n$ and see that ${}^t\mathbf{v}\,{}^tA^{\frac{1}{2}}BA^{\frac{1}{2}}\mathbf{v} = {}^t(A^{\frac{1}{2}}\mathbf{v})B(A^{\frac{1}{2}}\mathbf{v}) \geq 0$ since B is positive semi-definite. Thus $\sigma(AB) = \sigma({}^tA^{\frac{1}{2}}BA^{\frac{1}{2}}) \geq 0$. □