

INTRODUCTION AND MOTIVATION

Let F be some $m \times m$ positive definite matrix with integer entries and even diagonal.

Definition 1. The representation number [2] of $g \in \mathbb{N}$ by F is defined as the number of ways to represent g by F :

$$r_F(g) := \#\{\mathbf{x} \in \mathbb{Z}^m : {}^t\mathbf{x}F\mathbf{x} = g\} \quad (1)$$

We might ask whether we can find some formulae for calculating these numbers. We can use a type of modular form called a theta series to answer this question.

Definition 2. The Theta series associated with F is:

$$\theta_F(\tau) := \sum_{\mathbf{x} \in \mathbb{Z}^m} e^{\pi i ({}^t\mathbf{x}F\mathbf{x})\tau} = \sum_{g=0}^{\infty} r_F(g)q^g \quad (2)$$

On the right we have let $q = e^{2\pi i\tau}$ and rearranged the series to find that its Fourier coefficients are the representation numbers (1). We can then find bounds or explicit formulae for the $r_F(g)$ by applying results from the theory of modular forms.

A natural question to ask is whether we can find some automorphic forms with the following, more general, representation numbers as Fourier coefficients:

Definition 3. Let $G \in M_n(\mathbb{Z})$, F as defined above. The representation number [1] of G by F is:

$$r_F(G) := \#\{x \in M_{m \times n}(\mathbb{Z}) : {}^t x F x = G\} \quad (3)$$

Siegel Modular Forms provide us with an answer to this question.

DEFINITIONS

We will define a number of notions [4] which will provide the setting for Siegel Modular Forms. Setting $n = 1$ below gives the case of classical modular forms.

Definition 4.

- Siegel's half-space
 $\mathbb{H}_n := \{z = x + iy : z \in M_n(\mathbb{C}), z = {}^t z, y > 0\}$
where $y > 0$ here means that y is positive definite.
- The Symplectic Group
 $\text{Sp}(n, \mathbb{R}) := \{\gamma \in \text{GL}(2n, \mathbb{R}) : {}^t \gamma j \gamma = j\}$
where $j = \begin{pmatrix} 0_{n \times n} & 1_{n \times n} \\ -1_{n \times n} & 0_{n \times n} \end{pmatrix}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
with $a, b, c, d \in M_n(\mathbb{R})$
- The group action $\gamma(z) := (az + b)(cz + d)^{-1}$
- The modular group $\Gamma_n := \text{Sp}(n, \mathbb{Z})$
- Congruence subgroups of Γ_n of level q
 $\Gamma_0^{(n)}(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(n, \mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$

VISUALISING SIEGEL'S HALF-SPACE OF DEGREE 2: \mathbb{H}_2

Visualising \mathbb{H}_n can be challenging. Setting $n = 1$ just gives the complex upper half-plane. The first non-trivial case is $n = 2$. Points in \mathbb{H}_2 can be written as:

$$z = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + i \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \quad (4)$$

where we also impose the conditions $y_1 > 0$ and $y_1 y_3 - y_2^2 > 0$ which follow from y being positive definite. We see that the real part with coordinates (x_1, x_2, x_3) can be viewed as a copy of \mathbb{R}^3 , whilst the imaginary part with coordinates (y_1, y_2, y_3) can be viewed as a 'cone' sitting in a separate copy of \mathbb{R}^3 .

Essentially, \mathbb{H}_2 can be viewed as \mathbb{R}^3 with the cone in the figure 'attached' at every point to give a space with 6 total real dimensions. This notion is generalisable to \mathbb{H}_n .

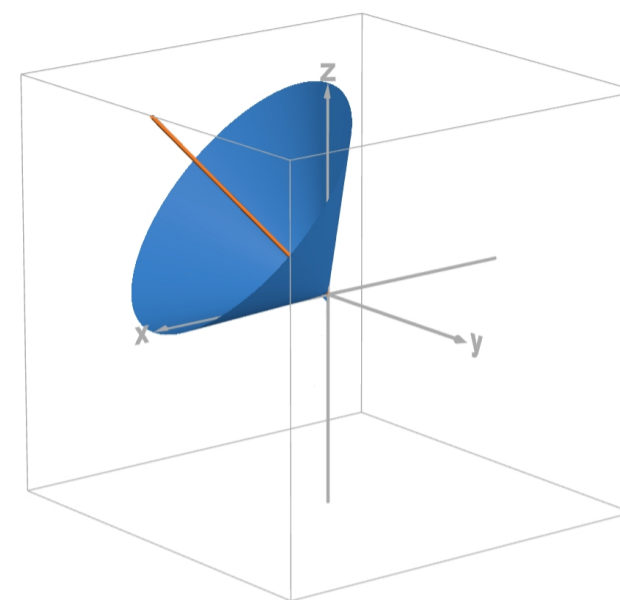


Figure 1: The boundary of the imaginary part of \mathbb{H}_2

SIEGEL MODULAR FORMS

We are now able to define Siegel Modular Forms. Our definition [2] will be more general than those found in common textbooks [4] to account for the fact that we will need to consider congruence subgroups for the study of Theta series.

Definition 5. A Siegel Modular Form [2][4] of degree n , weight k and level q is a function $f: \mathbb{H}_n \rightarrow \mathbb{C}$ such that:

- f is holomorphic
- f satisfies the transformation law:

$$f(\gamma(z)) = \chi(\gamma) \det(cz + d)^k f(z) \quad (5)$$

for all $\gamma \in \Gamma_0^{(n)}(q)$, $z \in \mathbb{H}_n$, where $\chi(\gamma) = \chi(\det(d))$ is some Dirichlet character modulo q .

- f is holomorphic at all cusps of $\Gamma_0^{(n)}(q)$. (Only when $n = 1$, i.e. the classical case. Otherwise this follows from (i) and (ii) by the Koecher principle [4])

Remark 6. The Siegel Modular Forms of degree n , weight k , level q and character χ form a vector space over \mathbb{C} . Thus, it can be shown that many of the structure theory results from classical modular forms can be extended to the Siegel case.

REFERENCES

- [1] A. N. Andrianov and G. N. Maloletkin. "Behaviour of Theta Series of degree N under modular substitutions." In: *Mathematics of the USSR-Izvestiya* 9.2 (1975), pp. 227–241.
- [2] Martin Dickson. "On Siegel Modular Forms On $\Gamma_0(N)$ ". PhD thesis. School of Mathematics, University of Bristol, 2015.
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THETA SERIES ARE MODULAR FORMS

Andrianov and Maloletkin [1] proved the following theorem:

Theorem 9.

$$\theta_F^{(n)}(\gamma(z)) = \chi_F^{(n)}(\gamma) \det(cz + d)^{\frac{m}{2}} \theta_F^{(n)}(z) \quad (8)$$

for all $\gamma \in \Gamma_0^{(n)}(q)$, $z \in \mathbb{H}_n$, where q is the level of the matrix F and $\chi_F^{(n)}(\gamma)$ is some eighth root of unity.

Since our theta series (6) are clearly holomorphic, it follows that the theta series $\theta_F^{(n)}(z)$ is a Siegel Modular Form of degree n , weight $m/2$ and level q . Furthermore, Andrianov and Maloletkin determine the character $\chi_F^{(n)}(\gamma)$ for even m . For odd m , not much is known about the character.

Remark 10. Theta series being Siegel modular forms means we may be able to apply known results to write Theta series in terms of Eisenstein series. From this we could find a way to write the representation numbers $r_F(G)$ in terms of Fourier coefficients of Eisenstein series, which are well understood.

We will now briefly discuss the strategy for proving theorem 9:

Proof.

1. Eichler [3] proved that a similar functional equation to (8) holds for the 'Theta Function':

$$\vartheta_r(\tilde{z}) := \sum_{\tilde{\mathbf{x}} \in \mathbb{Z}^r} \exp(\pi i {}^t \tilde{\mathbf{x}} \tilde{z} \tilde{\mathbf{x}}), \tilde{z} \in \mathbb{H}_r \quad (9)$$

2. We represent the theta series $\theta_F^{(n)}(z)$ as the value of the theta function $\vartheta_{mn}(\tilde{z})$ for a suitable choice of \tilde{z} . [1]
3. Using tensor products, define $\tilde{z} := F \otimes z$. It can be shown that $\tilde{z} \in \mathbb{H}_{mn}$.
4. For $x \in M_{m \times n}(\mathbb{Z})$ define $\tilde{\mathbf{x}} \in \mathbb{Z}^{mn}$ as the columns of x 'stacked' vertically into one long column vector.
5. This allows us to show

$$\begin{aligned} \theta_F^{(n)}(z) &= \sum_{x \in M_{m \times n}(\mathbb{Z})} \exp(\pi i \sigma(z {}^t x F x)) \\ &= \sum_{\tilde{\mathbf{x}} \in \mathbb{Z}^{mn}} \exp(\pi i {}^t \tilde{\mathbf{x}} \tilde{z} \tilde{\mathbf{x}}) = \vartheta_{mn}(\tilde{z}) \end{aligned}$$

6. Finally, we set $\tilde{a} := 1_m \otimes a$, $\tilde{b} := F \otimes b$, $\tilde{c} := F^{-1} \otimes c$ and $\tilde{d} := 1_m \otimes d$ to embed $\Gamma_0^{(n)}(q)$ in the equivalent modular group for the theta function. This then gives the functional equation (8).

□